Balanced Derivatives, Identities, and Bounds for Trigonometric Sums and Bessel Series

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Co-Authors

Joint Work with

Martino Fassina
Sun Kim
Alexandru Zaharescu

Martino Fassina

Figure: Martino Fassina, Saint Anthony Basilica,Padova

Sun Kim

Figure: Sun Kim

Alexandru Zaharescu

Figure: Alexandru Zaharescu

COVID-19

**Our thoughts are with the millions of families who are suffering from economic difficulties,
serious illness,
and death due to covid-19.**

Ramanujan's Passport Picture

Figure: Ramanujan

Quote from Ramanujan's First Letter to Hardy

Page 3, Item (4)

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, ... are numbers which are either themselves squares or which can be expressed as the sum of two squares.
The number of such numbers greater than A and less than B

$$= K \int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x) \tag{1}$$

where $K = .764$ and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found though complicated.

($\theta(x)$ should be replaced by $\theta(B)$.)

Quote from Hardy

The dominant term, viz. $KB(\log B)^{-1/2}$*, in Rammanujan's notation, was first obtained by Landau in 1908. Ramanujan had none of Landau's weapons at his command; . . . It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve . . .*

G. H. Hardy
Collected Papers of Srinivasa Ramanujan, p. xxiv

G. H. Hardy and J. E. Littlewood

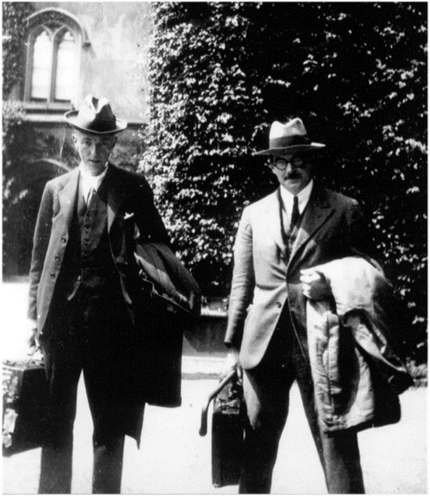

Figure: G. H. Hardy and J. E. Littlewood

Appearances in Ramanujan's Notebooks

page 307 of Ramanujan's second notebook.

The no. sum of two squares between A and B

$$= C \int_A^B \frac{dx}{\sqrt{\log x}} \quad \textit{nearly} \qquad \textit{where} \quad C = .764$$

$$C = \frac{1}{\sqrt{2(1 \quad)(1 \quad)(1 \quad)(1 \quad)\&c}} \qquad 3, 7$$

(Underneath the last equality sign appears: 9 7 ϵ[?].) Indeed, Ramanujan did not specify the expressions within his parentheses.

Appearances in Ramanujan's Notebooks

page 350 of Ramanujan's third notebook.

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- More space (the entire page) is devoted to his proof than any other argument or proof in the notebooks.
- The third notebook may not have been available to Hardy and Watson. Watson's handwritten personal copy of the notebooks does not contain the third notebook.

Circle Problem

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Different signs and different orders of the summands yield distinct representations. E.g.,

$$5 = (\pm 2)^2 + (\pm 1)^2, \qquad r_2(5) = 8.$$

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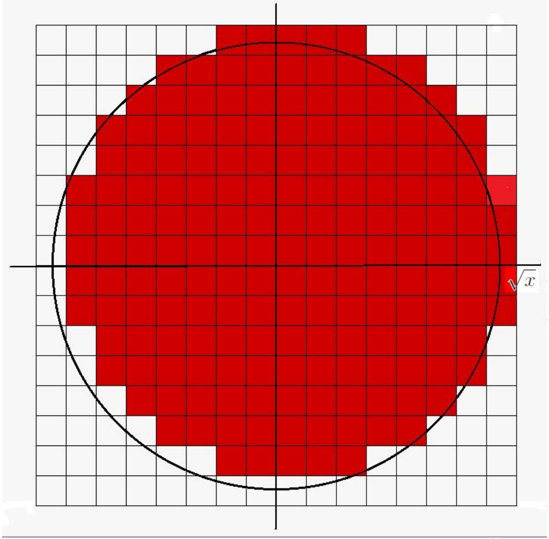
Each representation of n as a sum of two squares can be associated with a lattice point in the plane. For example, $5 = (-2)^2 + 1^2$ can be associated with the lattice point $(-2, 1)$. Then each lattice point can be associated with a unit square, say that unit square for which the lattice point is in the southwest corner.

Circle Problem



Circle Problem

$$R(x) := \sum_{0\le n\le x}{}' r_2(n) = \pi x + P(x), \tag{2}$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted.

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where the prime $\prime$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted.

$$R(x) < \pi(\sqrt{x}+\sqrt{2})^2,$$
$$R(x) > \pi(\sqrt{x}-\sqrt{2})^2,$$

$$R(x) = \pi x + O(\sqrt{x}) \qquad \text{Gauss}$$

Bounds for $P(x)$

The current best result is due to M. N. Huxley in 2003, namely, for every $\epsilon > 0$,

$$P(x) = O(x^{131/416+\epsilon}), \tag{3}$$

as $x \to \infty$. Note that

$$\frac{131}{416} = 0.3149\ldots.$$

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Conjecture: For every $\epsilon > 0$,

$$P(x) = O(x^{1/4+\epsilon}), \quad x \to \infty.$$

Best Ω-result

G. H. Hardy, On the expression of a number as the sum of two squares, Quart. J. Math. (Oxford) **46** (1915), 263–283.

Circle Problem

Ramanujan (1914?) and Hardy (1915) proved that

$$\sum_{n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}). \tag{4}$$

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0<|z|<\infty, \qquad \nu\in\mathbb{C}.$$

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"The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1) + d(2) + \cdots + d(n)$, where $d(n)$ is the number of divisors of n."

Another Beautiful Identity of Ramanujan as Recorded by Hardy

If $a, b > 0$, then

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}}, \qquad (5)$$

which is not given anywhere in Ramanujan's work.

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If we differentiate (5) with respect to b, let $a \to 0$, replace $2\pi\sqrt{b}$ by s, and use analytic continuation, we find that, for Re $s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2 + 4\pi^2 n)^{3/2}},$$

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which was the key identity in Hardy's proof of

$$P(x) = \Omega_{\pm}(x^{1/4}), \qquad \text{as } x \to \infty.$$

An Elementary Formula

Identity of Jacobi

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$$\begin{aligned}
\sum_{0<n\leq x}{}' r_2(n) &= 4 \sum_{0<n\leq x}{}' \sum_{d|n} \sin\left(\frac{\pi d}{2}\right) \\
&= 4 \sum_{0<dj\leq x}{}' \sin\left(\frac{\pi d}{2}\right) \\
&= 4 \sum_{0<d\leq x}{}' \left[\frac{x}{d}\right] \sin\left(\frac{\pi d}{2}\right),
\end{aligned}$$

where $[x]$ is the greatest integer $\leq x$.

An identity involving $r_2(n)$ found in a one-page manuscript published with Ramanujan's Lost Notebook. p. 335

$$0 < \theta < 1.$$

$$\left[\frac{x}{1}\right]\sin 2\pi\theta + \left[\frac{x}{2}\right]\sin 4\pi\theta + \left[\frac{x}{3}\right]\sin 6\pi\theta + \left[\frac{x}{4}\right]\sin 8\pi\theta + \cdots$$

$$= \pi x\left(\tfrac{1}{2}-\theta\right) - \tfrac{1}{2}\cot\pi\theta + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{J_1(4\pi\sqrt{\lambda\theta x})}{\sqrt{\lambda\theta}} - \frac{J_1(4\pi\sqrt{\lambda(1-\theta)x})}{\sqrt{\lambda(1-\theta)}} + \frac{J_1(4\pi\sqrt{\lambda(1+\theta)x})}{\sqrt{\lambda(1+\theta)}} - \frac{J_1(4\pi\sqrt{\lambda(2-\theta)x})}{\sqrt{\lambda(2-\theta)}} + \frac{J_1(4\pi\sqrt{\lambda(2+\theta)x})}{\sqrt{\lambda(2+\theta)}} - \cdots\cdots\right\}$$

where $[x]$ denotes the greatest integer in x if x is not an integer and $x-\frac{1}{2}$ if x is an integer.

$$\left[\frac{x}{1}\right]\cos 2\pi\theta + \left[\frac{x}{2}\right]\cos 4\pi\theta + \left[\frac{x}{3}\right]\cos 6\pi\theta + \left[\frac{x}{4}\right]\cos 8\pi\theta + \cdots$$

$$= -x\log(2\sin\pi\theta) + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{I_1(4\pi\sqrt{\lambda\theta x})}{\sqrt{\lambda\theta}} + \frac{I_1(4\pi\sqrt{\lambda(1-\theta)x})}{\sqrt{\lambda(1-\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(1+\theta)x})}{\sqrt{\lambda(1+\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(2-\theta)x})}{\sqrt{\lambda(2-\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(2+\theta)x})}{\sqrt{\lambda(2+\theta)}} + \cdots\cdots\right\},$$

where $I_1(x) = H_1(x) - Y_1(x)$.

The First Entry

Entry

Let $J_1(x)$ denote the ordinary Bessel function of order 1, let $0 < \theta < 1$, and let $x > 0$. Then

$$\sum_{n \leq x}{}' \left[\frac{x}{n}\right] \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta)$$

$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

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BCB, S. Kim and A. Zaharescu, *The circle and divisor problems, and double series of Bessel functions*, Adv. Math. **236** (2013), 24–59.

Sun Kim

Figure: Sun Kim at Graduation with Her Advisor

Version of BCB & Zaharescu

Theorem

Let $J_1(x)$ *denote the ordinary Bessel function of order* 1, *let* $0 < \theta < 1$, *and let* $x > 0$. *Then*

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BCB and A. Zaharescu, *Weighted divisor sums and Bessel function series*, Math. Ann. **335** (2006), 249–283.

Remarks

- Note the case $\theta = \frac{1}{4}$,

$$\sum_{0<n\leq x}{}' r_2(n) = 4 \sum_{0<d\leq x}{}' \left[\frac{x}{d}\right] \sin\left(\frac{\pi d}{2}\right)$$

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- Recall

$$\sum_{n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}).$$

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- Can we use the "extra" parameter θ to attack the *circle problem*?

Another Result from Ramanujan's First Letter to Hardy

(3) Let us take the number of divisors of natural numbers, viz. 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, ... (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, ...). The sum of such numbers to n terms

$$= n(2\gamma - 1 + \log n) + \frac{1}{2}\text{of the number of divisors of } n$$

where $\gamma = .5772156649\ldots$, the Eulerian Constant.

Dirichlet divisor problem

Let $d(n)$ denote the number of positive divisors of the positive integer n. Let

$$D(x) := \sum_{n \le x}{}' d(n),$$

where the prime $'$ indicates that if x is an integer, then we only count $\frac{1}{2}d(x)$. We see that

$$D(x) = \sum_{n\le x}{}' \sum_{d|n} 1 = \sum_{dj \le x}{}' 1 = \sum_{d \le x}{}' \sum_{1 \le j \le x/d} 1 = \sum_{d \le x}{}' \left[\frac{x}{d} \right],$$

where $[x]$ is the greatest integer less than or equal to x.

Geometric Interpretation

If $n = dj$, as above, we observe that n is uniquely associated with the lattice point (d, j) in the first quadrant under or on the hyperbola $ab = x$. Hence, $D(x)$ is equal to the number of lattice points in the first quadrant under or on the hyperbola $ab = x$.

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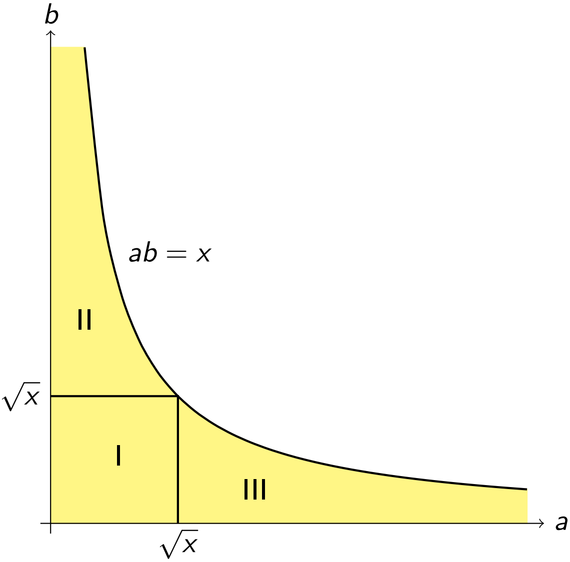
Dirichlet's divisor problem is equivalent to the problem of estimating the number of lattice points under or on a certain hyperbola.

Geometrical Interpretation



Dirichlet Divisor Problem

Theorem

For $x > 0$,

$$D(x) := {\sum_{n \le x}}' d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x), \qquad (6)$$

where γ is Euler's constant, and $\Delta(x)$ is the "error term." Then, as $x \to \infty$,

$$\Delta(x) = O(\sqrt{x}). \qquad (7)$$

The *Dirichlet divisor problem* asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$.

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z), \tag{8}$$

where $Y_\nu(z)$ denotes the Bessel function of imaginary argument of order ν given by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad |z| < \infty, \tag{9}$$

and $K_\nu(z)$ denotes the modified Bessel function of order ν defined by

$$K_\nu(z) := \frac{\pi}{2} \frac{e^{\pi i\nu/2} J_{-\nu}(iz) - e^{-\pi i\nu/2} J_\nu(iz)}{\sin(\nu\pi)}, \quad -\pi < \arg z < \tfrac{1}{2}\pi. \tag{10}$$

Voronoï's Formula

$$\sum_{n\le x}{}' d(n) = x\left(\log x + 2\gamma - 1\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{11}$$

where $x > 0$, γ denotes Euler's constant, and $I_1(z)$ is defined by

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{12}$$

Voronoï's Bound for $\Delta(x)$

In deriving the first improvement on Dirichlet's upper bound for $\Delta(x)$, in 1904, Voronoï proved that

$$\Delta(x) = O(x^{1/3} \log x), \qquad x \to \infty.$$

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The current best result is due to M. N. Huxley in 2003, namely, for every $\epsilon > 0$,

$$\Delta(x) = O(x^{131/416+\epsilon}),$$

as $x \to \infty$.

Second Identity

An identity involving $d(n)$ found in a one-page manuscript published with Ramanujan's Lost Notebook.
p. 335

Ramanujan's Second Identity

Entry (p. 335)

For $x > 0$ *and* $0 < \theta < 1$,

$$\sum_{n \le x}{}' \left[\frac{x}{n}\right] \cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) \tag{13}$$

$$+ \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\},$$

where

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z).$$

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- Note the left side when $\theta = 0$.

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- Our first theorem. If we invert the order of summation and assume convergence for one value of θ, then we can prove the identity for all values of θ.

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- BCB, S. Kim and A. Zaharescu, *Weighted divisor problems and Bessel function series, II*, Adv. Math. **229** (2012), 2055–2097.

The Final Problem

G. N. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. **11** (1936), 55–80.

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Sherlock Holmes' famous sidekick was Dr. Watson.

Final Problem for George Andrews and Myself

The identity involving the divisor function $d(n)$ on page 335 of Ramanujan's Lost Notebook.

This was OUR Final Problem.

Final Problem Has Been Solved

BCB, J. Li, and A. Zaharescu, *The final problem: an identity from Ramanujan's lost notebook*, J. London Math. Soc. **100** (2019), 568–591.

BCB, J. Li, and A. Zaharescu, *The Final Problem: A Series Identity From The Lost Notebook*, in: **George Andrews 80 Years of Combinatory Analysis** , K. Alladi, B. C. Berndt, P. Paule, J. Sellers, and A. J. Yee, eds., Birkhäuser, 2021, pp. 783–790.

Junxian Li and Alexandru Zaharescu

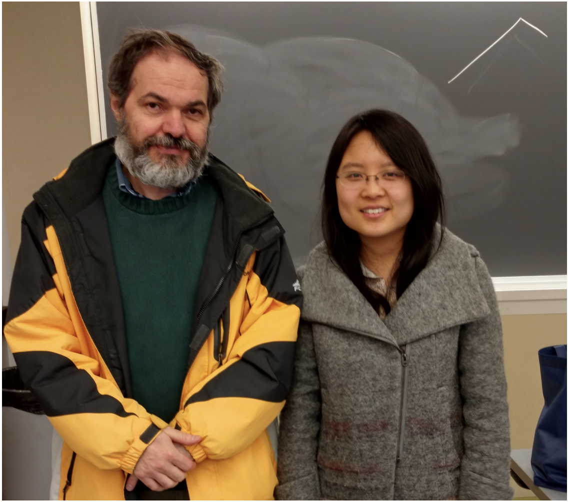

Figure: Alexandru Zaharescu and Junxian Li

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5. Must consider separately the case when x is an integer or not an integer.
6. Must divide the intervals $(0,\infty)$ for each summation variable in intervals for both "small" and "large" values of m and n.

Sketch of the Proof, Continued

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2. Isolate the series of Bessel functions on the right-hand side.
3. We calculate the Fourier series on both sides of the amended identity. We show that the Fourier series are identical.
4. Because both sides of our identity are continuous, we can appeal to the uniqueness theorem for Fourier series to conclude that the functions are identical.

Reference

BCB, S. Kim and A. Zaharescu, *Weighted divisor sums and Bessel function series, III*, J. Reine Angew. Math. **683** (2013), 67–96.

Trigonometric Sums

Let

$$\mathbb{SS}(x) := \mathbb{SS}(\sigma, \theta; x) := \sum_{mn \le x}{}' mn \sin(2\pi m\sigma) \sin(2\pi n\theta). \qquad (14)$$

Conjecture for $\mathbb{SS}(\sigma, \theta; x)$

Conjecture

If $\mathbb{SS}(\sigma, \theta; x)$ *is defined by* (14). *Then, for every* $\epsilon > 0$, *as* $x \to \infty$,

$$\mathbb{SS}(\sigma, \theta; x) = \Omega_{\pm}(x^{5/4})$$

Theorem for $\mathbb{SS}(\sigma, \theta; x)$

Theorem

As $x \to \infty$, for every $\epsilon > 0$,

$$\mathbb{SS}(\sigma, \theta; x) = O(x^{4/3+\epsilon}).$$

$$\frac{4}{3} = 1.333\ldots, \qquad \frac{5}{4} = 1.25.$$

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Special Case: A Lattice Point Problem

Let $\theta = \sigma = \frac{1}{4}$. Then

$$\sin(2\pi n/4) = \begin{cases} (-1)^{(n-1)/2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Then

$$\begin{aligned} \mathbb{SS}(\tfrac{1}{4}, \tfrac{1}{4}; x) &= -\sum_{\substack{mn \le x \\ m,n \text{ odd}}} mn(-1)^{(m+n)/2} \\ &= \sum_{(2j+1)(2k+1) \le x} (-1)^{j+k}(2j+1)(2k+1), \end{aligned} \tag{15}$$

where we set $m = 2j+1$, $n = 2k+1$. This is a rather interesting lattice point problem. We are counting lattice points under the hyperbola $ab \le x$, but we require both coordinates to be odd and we put a weight on them.

Multiple Sine Sums

$$\mathbb{S}(a_1, a_2, \dots a_k; p_1, p_2, \dots p_k; x) :=$$

$$\sum_{1 \le n_1 n_2 \cdots n_k \le x}{}' n_1 n_2 \cdots n_k \; \sin(2\pi n_1 a_1/p_1) \sin(2\pi n_2 a_2/p_2) \cdots \sin(2\pi n_k a_k/p_k)$$

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Conjecture

$$\mathbb{S}(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) = \Omega(x^{(3k-1)/(2k)})$$

Big O Theorem

Theorem

For every $\epsilon > 0$, *as* $x \to \infty$,

$$\mathbb{S}(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) = O\left(x^{2k/(k+1)+\epsilon}\right). \qquad (16)$$

Note that

$$\frac{2(k+1)}{k+2} - \frac{2k}{k+1} = \frac{2}{(k+1)(k+2)}$$

is the difference in the exponents of (16) for successive values of k. Thus, increasing the number of sin's by 1 in $\mathbb{S}(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)$ increases the upper bound for the power in the error term by a "small" amount, i.e., $O(1/k^2)$.

Another Remark on Big O

$$\frac{2k}{k+1} - \frac{3k-1}{2k} = \frac{(k-1)^2}{2k(k+1)} = \frac{1}{12}$$

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$$\frac{2k}{k+1} - \frac{3k-1}{2k} = \frac{(k-1)^2}{2k(k+1)} = \frac{1}{12}$$

$$k = 2, \qquad \frac{4}{3} - \frac{5}{4} = \frac{1}{12}$$

Derivatives

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If we have an unbalanced number of derivatives, we are unable to establish convergence.

First "Balanced" Result

Theorem

Let σ, θ be in the interval $(0,1)$, and let $x > 0$. Then for every non-negative integer k,

$$\frac{\partial^{2k}}{\partial\sigma^k \partial\theta^k}\left\{ {\sum_{mn\le x}}' \cos(2\pi m\sigma)\sin(2\pi n\theta) + \frac{\cot(\pi\theta)}{4} \right\}$$

$$= \frac{\sqrt{x}}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial^{2k}}{\partial\sigma^k \partial\theta^k}$$

$$\left\{ \frac{J_1(4\pi\sqrt{(m+\sigma)(n+\theta)x})}{\sqrt{(m+\sigma)(n+\theta)}} + \frac{J_1(4\pi\sqrt{(m+1-\sigma)(n+\theta)x})}{\sqrt{(m+1-\sigma)(n+\theta)}} \right.$$

$$\left. - \frac{J_1(4\pi\sqrt{(m+\sigma)(n+1-\theta)x})}{\sqrt{(m+\sigma)(n+1-\theta)}} - \frac{J_1(4\pi\sqrt{(m+1-\sigma)(n+1-\theta)x})}{\sqrt{(m+1-\sigma)(n+1-\theta)}} \right\}.$$

Key Series

$$G_\nu(x,\sigma,\theta,s,w) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{J_\nu(4\pi\sqrt{(m+\sigma)(n+\theta)x})}{(m+\sigma)^s(n+\theta)^w}\right.$$
$$\pm\frac{J_\nu(4\pi\sqrt{(m+1-\sigma)(n+\theta)x})}{(m+1-\sigma)^s(n+\theta)^w}\pm\frac{J_\nu(4\pi\sqrt{(m+\sigma)(n+1-\theta)x})}{(m+\sigma)^s(n+1-\theta)^w}$$
$$\left.\pm\frac{J_\nu(4\pi\sqrt{(m+1-\sigma)(n+1-\theta)x})}{(m+1-\sigma)^s(n+1-\theta)^w}\right), \tag{17}$$

Key Theorem

Theorem

Let $G_\nu(x,\sigma,\theta,s,w)$ be defined as above. Assume that $4\,\mathrm{Re}(s) > 1$, and that $4\,\mathrm{Re}(w) > 1$. Moreover, if x is an integer, assume $\mathrm{Re}(s) + \mathrm{Re}(w) > \frac{25}{26}$, while if x is not an integer, assume $\mathrm{Re}(s) + \mathrm{Re}(w) > \frac{5}{6}$. Then the double series $G_\nu(x,\sigma,\theta,s,w)$ converges uniformly with respect to σ and θ in any compact subset of $(0,1)^2$.

Cosine-Cosine Sum

Theorem

Let $I_1(x)$ *be defined by* (8)*. If* $0 < \theta,\ \sigma < 1$ *and* $x > 0$*, then*

$$\sum_{nm\leq x}{}' \cos(2\pi n\theta)\cos(2\pi m\sigma)$$

$$= \frac{1}{4} + \frac{\sqrt{x}}{4}$$

$$\sum_{n,m\geq 0}\left\{\frac{I_1(4\pi\sqrt{(n+\theta)(m+\sigma)x})}{\sqrt{(n+\theta)(m+\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+\sigma)x})}{\sqrt{(n+1-\theta)(m+\sigma)}}\right.$$

$$\left. + \frac{I_1(4\pi\sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}}\right\}$$

Further Motivation

It is sufficient to prove each of Ramanujan's formulas on page 335 in his Lost Notebook with one order of summation

Ramanujan's Home, Kumbakonam

Ramanujan's Home, Porch

I am very grateful for the invitation.

Thank You