Parity Bias in Partitions

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Two Goals

1. Describe the results in:

   B. Kim, E. Kim, and J. Lovejoy, *Parity Bias in Partitions*, European. J. Combin. 89 (2020).

2. Discuss some conjectures reminiscent of conjectures of Andrews on certain q-series in Ramanujan's lost notebook.

Introduction

Partition-theoretic generating functions are often related to number theoretic objects, and in this case one can use number theory to establish facts about partitions.

For example, let $p(n)$ denote the number of partitions of n.

Ramanujan showed that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

and wrote that "it appears that there are no equally simple properties for any moduli involving primes other than these three."

Introduction (cont.)

Using the fact that

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}$$

is essentially a weakly holomorphic modular form, Ahlgren and Boylan (2003) confirmed Ramanujan's observation.

Introduction (cont.)

For another example, let $S(n)$ denote the number of partitions of n into distinct parts with even rank minus the number of such partitions with odd rank.

The rank of a partition is the largest part minus the number of parts.

Andrews (1986) conjectured that

1) $\limsup |S(n)| = \infty$,
2) $S(n) = 0$ for infinitely many n.

Introduction (cont.)

Using the fact that

$$\sum_{n\geq 0} S(n)q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{\prod_{k=1}^{n}(1+q^k)}$$
$$= \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}),$$

Andrews, Dyson, and Hickerson (1988) confirmed Andrews' conjectures (and much more) using the arithmetic of the real quadratic field $\mathbb{Q}(\sqrt{6})$.

Introduction (cont.)

And sometimes partition-theoretic generating functions don't appear to be related to number-theoretic objects.

For example, let $q_d(n)$ denote the number of partitions of n into parts differing by at least d and let $Q_d(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod{d+3}$.

Alder conjectured that for all positive d and n,

$$q_d(n) \geq Q_d(n).$$

Introduction (cont.)

The generating function

$$\sum_{n\geq 0} Q_d(n)q^n = \prod_{n\geq 0} \frac{1}{(1-q^{(d+3)n+1})(1-q^{(d+3)n+(d+2)})}$$

is related to modular forms, but the generating function

$$\sum_{n\geq 0} \frac{q^{n+dn(n-1)/2}}{\prod_{k=1}^{n}(1-q^k)}$$

is not, in general.

Introduction (cont.)

The proof of Alder's conjecture ultimately required a combinatorial *tour de force* of Yee (2008) for the cases $d \geq 32$ and some precise asymptotic analysis of Afles, Jameson, and Lemke-Oliver for the remaining cases.

The generating functions related to parity bias appear unrelated to modular forms, mock modular forms, mixed mock modular forms, false theta functions,...

Parity Bias

Let $p_o(n)$ denote the number of partitions of n with more odd parts than even parts and let $p_e(n)$ denote the number of partitions of n with more even parts than odd parts.

It turns out that $p_o(n) > p_e(n)$ except for $n = 2$.

We refer to this as *parity bias* – the tendency of partitions to have more odd parts than even parts.

Example: $n = 6$

partition	$p_e(6)/p_o(6)$
(6)	$p_e(6)$
$(5,1)$	$p_o(6)$
$(4,2)$	$p_e(6)$
$(4,1,1)$	$p_o(6)$
$(3,3)$	$p_o(6)$
$(3,2,1)$	$p_o(6)$
$(3,1,1,1)$	$p_o(6)$
$(2,2,2)$	$p_e(6)$
$(2,2,1,1)$	–
$(2,1,1,1,1)$	$p_o(6)$
$(1,1,1,1,1,1)$	$p_o(6)$

We have $p_o(6) = 7$ and $p_e(6) = 3$.

Results

Theorem 1

For all $n \neq 2$, we have $p_o(n) > p_e(n)$.

Theorem 2

For all positive integers $n > 7$,

$$2p_e(n) < p_o(n) < 3p_e(n).$$

Theorem 3

As $n \to \infty$,

$$\frac{p_o(n)}{p_e(n)} \to 1 + \sqrt{2} \approx 2.4142.$$

Proofs

Theorem 1: Generating functions and q-series transformations.

Theorem 2: Combinatorial Mappings.

Theorem 3: Ingham's Theorem.

Notation

We use the usual q-series notation,

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

More on Theorem 1

Using standard combinatorial arguments, the term

$$\frac{q^{bn}}{(q^2;q^2)_n}$$

generates partitions into

$$\begin{cases} \text{at most } n \text{ even parts}, & \text{if } b = 0, \\ \text{exactly } n \text{ odd parts}, & \text{if } b = 1, \\ \text{exactly } n \text{ even parts}, & \text{if } b = 2. \end{cases}$$

More on Theorem 1 (cont.)

Therefore

$$\begin{aligned} P_o(q) &:= \sum_{n\geq 0} p_o(n)q^n \\ &= \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} \\ &= q + q^2 + 2q^3 + 3q^4 + 4q^5 + 7q^6 + 9q^7 + 14q^8 + \cdots, \end{aligned}$$

$$\begin{aligned} P_e(q) &:= \sum_{n\geq 0} p_e(n)q^n \\ &= \frac{1}{(q;q)_\infty} - \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2} \\ &= q^2 + 2q^4 + q^5 + 3q^6 + 3q^7 + 6q^8 + 7q^9 + 10q^{10} + \cdots \end{aligned}$$

More on Theorem 1 (cont.)

Naively subtracting these generating functions doesn't help.

But using classical q-series transformations, we obtain

$$P_o(q) = \frac{1}{(q;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2-n}}{(q^2;q^2)_n (q^2;q^2)_{n-1}}$$

and

$$P_e(q) = \frac{1}{(q;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2}(1-q^n)}{(q^2;q^2)_n^2}.$$

More on Theorem 1 (cont.)

Now subtracting gives

$$P_o(q) - P_e(q) = \frac{1}{(q;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2}.$$

Each summand clearly has non-negative coefficients.

For $n = 2$, the summand is

$$\frac{q^6}{(1-q^2)(1-q^4)^2(1-q)(q^3;q^2)_\infty},$$

which implies that for $n \geq 6$ the coefficient of q^n is positive.

The result follows.

Remarks

Classical q-series transformations are still very useful!

B. Kim and E. Kim have obtained similar biases in other arithmetic progressions (*Biases in integer partitions*, Bull. Aust. Math. Soc., to appear.)

The generating functions $P_o(q)$, $P_e(q)$, and $P_o(q) - P_e(q)$ appear unrelated to modular forms, mock modular forms, mixed mock modular forms, false theta functions,...

More on Theorem 2

Theorem 2 is shown in two steps.

1) $2p_e(n) < p_o(n)$ except for $n = 2, 4$.
2) $3p_e(n) > p_o(n)$ except for $n = 1, 3, 5, 7$.

Each part is done using an intricate injection.

We sketch the idea for the first part.

More on Theorem 2

Let A and B be sets of partitions.

Recall that a generalized Frobenius symbol of type (A, B) and weight n is a two-rowed array of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix},$$

where the top row $(a_1, a_2, \ldots, a_k)$ is a partition in the set A, the bottom row $(b_1, b_2, \ldots, b_k)$ is a partition in the set B, and $n = \sum_{i=1}^{k}(a_i + b_i + 1)$.

More on Theorem 2 (cont.)

Let $\mathcal{F}_{A,B}$ denote the set of Frobenius symbols of type (A, B) and let $F_{A,B}(n)$ to denote the number of Frobenius symbols in $\mathcal{F}_{A,B}$ having weight n.

Generalized Frobenius symbols can be used to represent a number of types of well-known partitions.

For example, let $\mathcal{D}$ be the set of partitions into distinct non-negative parts. Then

$$p(n) = F_{\mathcal{D},\mathcal{D}}(n).$$

For another example, let $\mathcal{O}$ be the set of overpartitions into non-negative parts. Then

$$\overline{p}(n) = F_{\mathcal{D},\mathcal{O}}(n),$$

More on Theorem 2 (cont.)

Let $\mathcal{D}_e$ denote the set of partitions into distinct non-negative even parts.

Let $\mathcal{A}$ be the set of partition pairs (π, λ), where π is a partition into distinct parts and λ is a Frobenius symbol in $\mathcal{F}_{\mathcal{D}_e,\mathcal{D}_e}$ such that 0 always occurs in the top row.

Let $\mathcal{B}$ be the set of partition pairs (π, λ) where λ is a Frobenius symbol in $\mathcal{F}_{\mathcal{D}_e,\mathcal{D}_e}$ such that 0 is always a part of the top row, the largest part on the top may be overlined, and π is a partition into distinct parts not equal to $\ell(\lambda)$.

Here $\ell(\lambda)$ denotes the number of columns in λ.

More on Theorem 2 (cont.)

It turns out that

$$\sum_{(\pi,\lambda)\in\mathcal{B}} q^{|\pi|+|\lambda|} - \sum_{(\pi,\lambda)\in\mathcal{A}} q^{|\pi|+|\lambda|}$$

$$= 2(-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2} - (-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}}{(q^2;q^2)_n (q^2;q^2)_{n-1}}$$

$$= 2\left(P_o(q) - P_e(q)\right) - P_o(q)$$

$$= P_o(q) - 2P_e(q)$$

$$= q - q^2 + 2q^3 - q^4 + 2q^5 + q^6 + \cdots + 82q^{20} + 107q^{21} + \cdots$$

To prove the desired inequality, we give an injection $\phi : \mathcal{A}_n \to \mathcal{B}_n$ for $n \neq 2, 4$, where the subscript n means $|\pi| + |\lambda| = n$.

More on Theorem 2 (cont.)

Suppose that $(\pi, \lambda) \in \mathcal{A}_n$. Define $\phi((\pi, \lambda)) = (\pi', \lambda') \in \mathcal{B}_n$ as follows:

Case 1: $\ell(\lambda)$ is not a part of π. Then $(\pi', \lambda') = (\pi, \lambda)$.

Case 2: $\ell(\lambda)$ is a part of π and π has at least two parts. Then λ' is the Frobenius symbol obtained by overlining the top leftmost part of λ and π' is constructed by deleting the part $\ell(\lambda)$ from the partition π and adding $\ell(\lambda)$ to the largest part among the remaining parts of π.

$$e.g.\ \left((3,2,1), \begin{pmatrix} 4 & 0 \\ 6 & 2 \end{pmatrix}\right) \to \left((5,1), \begin{pmatrix} \overline{4} & 0 \\ 6 & 2 \end{pmatrix}\right)$$

More on Theorem 2 (cont.)

Case 3: $\pi = (\ell(\lambda))$ and $\ell(\lambda) > 1$.

(i) If $\ell(\lambda)$ is even, λ' is obtained by adding $\ell(\lambda)$ to the top leftmost part of λ after overlining the top leftmost entry, and $\pi' = \emptyset$,

(ii) if $\ell(\lambda)$ is odd, λ' is obtained by adding $\ell(\lambda) - 1$ to the largest part of the top row of λ after overlining the top leftmost entry, and $\pi' = (1)$.

Case 4: $\pi = (\ell(\lambda))$ and $\ell(\lambda) = 1$. In this case, for $k > 1$ we define

$$\phi((\pi, \lambda)) = \phi\left(\left((1), \begin{pmatrix} 0 \\ 2k \end{pmatrix}\right)\right) = (\pi', \lambda') = \left(\emptyset, \begin{pmatrix} \overline{2} & 0 \\ 2k-2 & 0 \end{pmatrix}\right)$$

We then observe that:

1) ϕ is defined on all pairs $(\pi, \lambda) \in \mathcal{A}$ except for $k = 0, 1$ in Case 4, corresponding to $n = 2, 4$.

2) ϕ is an injection. This gives the inequality $p_o(n) \geq 2p_e(n)$ for $n \neq 2, 4$.

3) For $k \geq 2$, the pairs

$$\left((2,1), \begin{pmatrix} \overline{2k} & 2 & 0 \\ 4 & 2 & 0 \end{pmatrix}\right), \left((2), \begin{pmatrix} \overline{2k} & 2 & 0 \\ 4 & 2 & 0 \end{pmatrix}\right) \in \mathcal{B}$$

are not in $\phi(\mathcal{A})$. These have weight $n = 2k + 14$ and $2k + 13$, respectively, giving strict inequality for $n \geq 17$.

Remarks

The injection for the second inequality $3p_e(n) > p_o(n)$ is much more involved. (Multiple cases, subcases, and sub-subcases.)

We also have a q-series proof for $p_o(n) > 2p_e(n)$.

Note that we used the transformed generating functions and the corresponding pairs of partitions. Is it possible to argue directly using the partitions counted by $p_o(n)$ and $p_e(n)$?

More on Theorem 3

Some data:

n	$p_o(n)/p(n)$	$p_e(n)/p(n)$	$p_o(n)/p_e(n)$
100	0.6795	0.2764	2.4588
500	0.6946	0.2854	2.4339
1500	0.6998	0.2885	2.4255
2500	0.7015	0.2895	2.4229
5000	0.7031	0.2905	2.4204
10000	0.7043	0.2912	2.4186

We show that

$$p_o(n) \sim \frac{1}{\sqrt{2}} p(n) \approx 0.7071 p(n)$$

$$p_e(n) \sim \frac{\sqrt{2}-1}{\sqrt{2}} p(n) \approx 0.2929 p(n).$$

More on Theorem 3

We use classical asymptotic analysis (Ingham's Theorem).

First, we show that $p_o(n)$ and $p_e(n)$ are weakly increasing sequences. (The latter holds for $n > 5$.)

We have

$$(1-q)P_o(q) = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2-n}}{(q^2;q^2)_n (q^2;q^2)_{n-1}},$$

$$(1-q)P_e(q) = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2}(1-q^n)}{(q^2;q^2)_n^2}.$$

More on Theorem 3 (cont.)

The first series obviously has non-negative coefficients.

For $n \geq 2$ the nth summand of the second series has non-negative coefficients.

For $n = 1$ we use a q-series identity and obtain

$$\frac{q^2(1-q)^2}{(1-q^2)^2(q;q^2)_\infty} = -q^3 - q^5 + \frac{q^2(1+q^2)}{1-q^2} + \frac{q^2}{(1-q^2)} \sum_{n \geq 2} \frac{(-q^2)_{n-1}}{(q^2)_{n-1}} (1+q^{2n+1}) q^{(3n^2+n)/2}.$$

More on Theorem 3 (cont.)

Next we study $P_o(e^{-z})$ and $P_e(e^{-z})$ as $z \to 0$.

We show that

$$P_o(e^{-z}) \sim \frac{1}{2}\sqrt{\frac{z}{\pi}} \exp\left(\frac{\pi^2}{6z}\right),$$

$$P_e(e^{-z}) \sim \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\sqrt{\frac{z}{\pi}} \exp\left(\frac{\pi^2}{6z}\right),$$

which by Ingham's theorem gives

$$p_o(n) \sim \frac{1}{4\sqrt{6}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \frac{1}{\sqrt{2}} p(n),$$

$$p_e(n) \sim \frac{\sqrt{2}-1}{4\sqrt{6}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \frac{\sqrt{2}-1}{\sqrt{2}} p(n).$$

More on Theorem 3 (cont.)

To prove the asymptotics for $P_o(e^{-z})$ and $P_e(e^{-z})$, we use the fact that $P_o(q)$ and $P_e(q)$ can be written in terms of

$$G_b(q) = \frac{1}{(q;q^2)_\infty} \sum_{n \geq 0} \frac{q^{2n^2+bn}}{(q^2;q^2)_n^2}$$

for $b = 0, 1, -1$, and that $G_b(q)$ can be expressed as a constant term.

For example, using the usual Jacobi theta function $\theta(x;q)$,

$$G_1(q) = [x^0] \left(\frac{(-x;q)_\infty}{(q)_\infty (-x;q^2)_\infty^2} \theta(q; x/q) \right)$$

We then apply facts about theta functions and the quantum dilogarithm.

Further work and conjectures

It is natural to wonder about parity bias in other types of partitions.

For example, if $d_o(n)$ (resp. $d_e(n)$) denotes the number of partitions into distinct parts having more odd (resp. even) parts than even (resp. odd) parts, then it appears that

$$d_o(n) > d_e(n)$$

for $n > 19$. (This has been checked up to $n = 2000$.)

Solved by Dastidar, Bannerjee, Battacharjee (announced).

One can also ask about other kinds of biases...

Further work and conjectures (cont.)

Define $\overline{p}_u(n)$ (resp. $\overline{p}_o(n)$) to be the number of overpartitions of n with more non-overlined (resp. overlined) parts than overlined (resp. non-overlined) parts.

Theorem 4

The difference $\overline{p}_u(n) - \overline{p}_o(n)$ is equal to the number of overpartitions of n where the number of non-overlined parts is at least two more than the number of overlined parts.

For example, there are 8 overpartitions of 3,

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

Here $\overline{p}_u(3) = 4$, $\overline{p}_o(3) = 2$, and thus $\overline{p}_u(3) - \overline{p}_o(3) = 2$, which corresponds to the overpartitions $2+1$ and $1+1+1$.

Further work and conjectures (cont.)

It is natural to consider weighted versions of partition generating functions.

Think of:

$$f(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2},$$

$$\overline{f}(q) = \sum_{n \geq 0} \frac{q^{n(n+1)/2}(-1;q)_n}{(-q;q)_n^2},$$

$$\sigma(q) = \sum_{n \geq 0} \frac{q^{n(n+1)/2}}{(-q;q)_n}.$$

Further work and conjectures (cont.)

Consider the series

$$\sum_{n\geq 1} a(n)q^n = \sum_{n\geq 1} \frac{q^n(1+q^{2n})}{(-q^2;q^2)_n^2}.$$

It turns out that $a(n)$ is the number of partitions λ counted by $p_o(n)$, each weighted by $(-1)^w$, where

$$w = \frac{1}{2}(\text{the largest even part of } \lambda + \text{the largest odd part of } \lambda - 1).$$

Further work and conjectures (cont.)

It appears that:

– $a(n) = 0$ if and only if

$$n \in \{3, 5, 9, 17, 20, 23, 24, 26, 28, 51, 125, 233\}.$$

– $a(n)$ has a regular sign pattern $+, +, -, -$ with infinitely many (sporadic) exceptions.

We have no proof of this.

Further work and conjectures (cont.)

In *Questions and Conjectures in Partition Theory*, Amer. Math. Monthly **93** (1986), 708–711, George Andrews observed that some series in Ramanujan's lost notebook also had coefficients with "a lengthy sign change pattern that alters fairly infrequently" and made some precise conjectures about this.

For example,

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}$$

What is the explanation for these "almost regular" sign patterns?

Thanks!