Cologne Number Theory Seminar

Fourier interpolation from zeros of $\zeta(s)$

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Overview

1. The Riemann zeta function and the explicit formula
2. A new Fourier duality relation for zeros of $\zeta(s)$
3. Reconstruction problem for the Fourier transform
4. Proof outline

The Riemann zeta function

Definition

Additive definition:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \quad \operatorname{Re} s > 1\,.$$

Multiplicative definition:

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}\,, \quad \operatorname{Re} s > 1\,.$$

The additive definition shows that $\zeta(s)$ continues analytically to a meromorphic function in $\mathbb{C}$ with a simple pole at $s = 1$.
The multiplicative definition shows that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$.

Analytic continuation

From $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$ and $\frac{1}{e^x - 1} = \sum_{n \geq 1} e^{-nx}$ one gets

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \,, \qquad \operatorname{Re} s > 1$$

This leads to analytic continuation by the following property of the Mellin transform

Proposition

Let $f : (0, \infty) \to \mathbb{C}$ be a continuous rapidly decaying function such that for some $c_j, \alpha_j \in \mathbb{C}$ with $\operatorname{Re} \alpha_j \to +\infty$ one has

$$f(x) \sim c_1 x^{\alpha_1} + c_2 x^{\alpha_2} + \dots \,, \qquad x \to 0.$$

Then $F(s) = \mathcal{M}(f)(s) := \int_0^\infty f(x) x^{s-1} dx$ continues analytically to a meromorphic function in $\mathbb{C}$ with simple poles at $s = -\alpha_j$ and $\operatorname{Res}_{s=-\alpha_j} F(s) = c_j$.

Analytic continuation

Alternatively, one can write

$$\Gamma_{\mathbb{R}}(s)\zeta(s) = \int_0^\infty \omega(x) x^{s/2-1} dx$$

Here

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)\,, \qquad \omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}\,.$$

The key observation is that

$$\omega(x) - x^{-1/2}\omega(1/x) = (x^{-1/2} - 1)/2$$

which implies

$$\Gamma_{\mathbb{R}}(1-s)\zeta(1-s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$$

Explicit formula

Note that the Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ implies that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p\,, & n = p^k \\ 0\,, & \text{otherwise} \end{cases}$$

The only poles of $\zeta'(s)/\zeta(s)$ in $-\varepsilon < \operatorname{Re} s < 1 + \varepsilon$ are at $s = 0, 1$, and at $s = \rho$ for each nontrivial zero ρ of $\zeta(s)$. The residue at $s = \rho$ is m_ρ, the multiplicity of ρ.

Explicit formula

Note that the functional equation implies

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} = -\frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)} - \frac{\Gamma'_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(1-s)}$$

Let h be an analytic function in the strip $-2\varepsilon < \operatorname{Re} s < 1 + 2\varepsilon$ satisfying $h(z) = h(1-z)$. Then we can calculate

$$I = \frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} h(s)\Big(\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)}\Big) ds$$

in two different ways:

$$I = \frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} h(s)\frac{\zeta'(s)}{\zeta(s)} ds - \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} h(s)\frac{\zeta'(s)}{\zeta(s)} ds$$

$$I = \frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} h(s)\Big(2\frac{\zeta'(s)}{\zeta(s)} + \frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)} + \frac{\Gamma'_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(1-s)}\Big) ds$$

Explicit formula

The first expression can be evaluated using Cauchy's theorem, and the second using the Mellin inversion formula

$$\mathcal{M}^{-1}(h)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(s)x^{-s}ds$$

Theorem (Riemann-Weil explicit formula)

Let $f(z)$ be an even function, analytic in $S_{1/2+\varepsilon} = \{z\colon |\operatorname{Im} z| < 1/2+\varepsilon\}$ and suppose that $|f(x+iy)| \ll (1+|x|)^{-1-\varepsilon}$. Then

$$f(\tfrac{i}{2}) + \frac{1}{2\pi}\int_{\mathbb{R}} f(t)\Psi(t)dt = \sum_{\substack{\zeta(\rho)=0 \\ \operatorname{Im}\rho>0}} m_\rho f\Big(\frac{\rho-1/2}{i}\Big) + \frac{1}{2\pi}\sum_{n\geq 1}\frac{\Lambda(n)}{n^{1/2}}\widehat{f}\Big(\frac{\log n}{2\pi}\Big)$$

Here $\Psi(t) = \frac{\Gamma'_{\mathbb{R}}(1/2+it)}{\Gamma_{\mathbb{R}}(1/2+it)}$, m_ρ is the multiplicity of ρ, and $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx$.

Explicit formula

One may view the explicit formula as an expression of the linear functional

$$f \mapsto f(\tfrac{i}{2}) + \frac{1}{2\pi}\int_{\mathbb{R}} f(t)\Psi(t)dt$$

as a linear combination of

$$f \mapsto f\Big(\frac{\rho - 1/2}{i}\Big) \quad (\zeta(\rho) = 0)\,, \qquad f \mapsto \widehat{f}\Big(\frac{\log n}{2\pi}\Big) \quad (n = p^k)$$

Roughly speaking, our main result is that essentially **any** continuous linear functional can be written as a linear combination of

$$f \mapsto f\Big(\frac{\rho - 1/2}{i}\Big)\,, \qquad f \mapsto \widehat{f}\Big(\frac{\log n}{4\pi}\Big)$$

Interpolation from zeros of $\zeta(s)$

Let $\mathcal{H}_1$ be the space of $f(z)$ analytic in the strip $|\operatorname{Im} z| < 1/2 + \varepsilon$ such that

$$\sup_{|y|<1/2+\varepsilon} \int_{-\infty}^{\infty} |f(x+iy)|(1+|x|)dx < \infty$$

Theorem (Bondarenko-R.-Seip, 2020)

There exist two sequences of rapidly decaying even entire functions $U_n(z)$, $n = 1, 2, \ldots$, and $V_{\rho,j}(z)$, $0 \le j < m_\rho$, with ρ ranging over the nontrivial zeros of $\zeta(s)$ such that for every even function f in $\mathcal{H}_1$ and z with $|\operatorname{Im} z| < 1/2$

$$f(z) = \sum_{n=1}^{\infty} \widehat{f}\left(\frac{\log n}{4\pi}\right) U_n(z) + \lim_{m\to\infty} \sum_{0<\gamma\le T_m} \sum_{j=0}^{m_\rho - 1} f^{(j)}\left(\frac{\rho - 1/2}{i}\right) V_{\rho,j}(z)$$

for some universal sequence of positive numbers $T_m \to \infty$.

Interpolation from zeros of $\zeta(s)$

Corollary

If an even function $f \in \mathcal{H}_1$ satisfies

$$f^{(j)}\Big(\frac{\rho - 1/2}{i}\Big) = 0\,, \qquad 0 \le j < m_\rho$$
$$\widehat{f}\Big(\frac{\log n}{4\pi}\Big) = 0\,, \qquad n \ge 1$$

then f must vanish identically.

This is optimal, since by construction U_n and $V_{\rho,j}$ satisfy

$$\begin{array}{rclrcl} U_n^{(j)}\left(\frac{\rho-1/2}{i}\right) & = & 0, & \widehat{U}_n\left(\frac{\log n'}{4\pi}\right) & = & \delta_{n,n'} \\ V_{\rho,j}^{(j')}\left(\frac{\rho'-1/2}{i}\right) & = & \delta_{(\rho,j),(\rho',j')}, & \widehat{V}_{\rho,j}\left(\frac{\log n}{4\pi}\right) & = & 0 \end{array}$$

Reconstruction problem for the Fourier transform

We use the following normalization for the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx$$

Question

How to recover a nice function $f : \mathbb{R} \to \mathbb{C}$ from partial information about f and $\widehat{f}$?

Specifically, given closed sets $A, B \subset \mathbb{R}$, how to recover f from $f \mapsto (f|_A, \widehat{f}|_B)$?

- If f is uniquely determined by $(f|_A, \widehat{f}|_B)$, (A, B) is a **(Fourier) uniqueness pair**
- We call a uniqueness pair **tight** if one cannot replace A, B by proper subsets
- Finally, in an ideal situation we may ask whether there is an explicit formula that recovers f from $f|_A$ and $\widehat{f}|_B$

Chebotarev-Tao theorem

Exmaple. Let p be a prime and for $f\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ define

$$\widehat{f}(\xi) = \frac{1}{\sqrt{p}} \sum_{x \pmod p} f(x) e^{-2\pi i x \xi / p}$$

Theorem (Tao, 2005)

A nonzero function $f\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ satisfies $|S(f)| + |S(\widehat{f})| \geq p+1$.
Here $S(g)$ denotes the support of a function $g\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$.

This result follows from an old theorem of Chebotarev from 1926 that all minors of the matrix $(e^{2\pi i k l / p})_{k,l}$ are nonzero.

Corollary

If $A, B \subset \mathbb{Z}/p\mathbb{Z}$ are such that $|A| + |B| = p$, then (A, B) is a tight uniqueness pair.

Whittaker-Shannon interpolation formula

Example. If $A = \{x\colon |x| \geq 1/2\}$, $B = \mathbb{Z}$, then (A, B) is a uniqueness pair.

Proof.

This follows from the Poisson summation formula

$$f(x) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) e^{2\pi i m x} - \sum_{n\neq 0} f(x+n)$$

□

If $f|_{|x|\geq 1/2} = 0$, taking the Fourier transform gives the Whittaker-Shannon formula:

$$\widehat{f}(x) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) \frac{\sin(\pi(x-m))}{\pi(x-m)}$$

Interpolation from $\pm\sqrt{n}$

Example. $A = \{\pm\sqrt{n}\}_{n\geq 0}$, $B = A \setminus \{0\}$, is a uniqueness pair for even functions.

Theorem (R.-Viazovska, 2017)

There exists a sequence of even Schwartz functions $a_n\colon \mathbb{R} \to \mathbb{R}$ with the property that for every even Schwartz function $f : \mathbb{R} \to \mathbb{R}$ we have

$$f(x) = \sum_{n=0}^{\infty} f(\sqrt{n})a_n(x) + \sum_{n=0}^{\infty} \widehat{f}(\sqrt{n})\widehat{a}_n(x)\,.$$

The value $\widehat{f}(0)$ can be removed from RHS using $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$.

Tightness follows from $a_n(\sqrt{m}) = \delta_{m,n}$, $\widehat{a_n}(\sqrt{m}) = 0$ for $n, m \geq 1$.

There is an analogous statement for odd functions that also involves $f'(0)$ and $\widehat{f}'(0)$.

Application: Universal optimality of lattices

Theorem (Cohn-Kumar-Miller-R.-Viazovska, 2019)

For $d \in \{8, 24\}$ there exist two sequences of radial Schwartz functions $a_n, b_n \in \mathcal{S}(\mathbb{R}^d)$, $n \geq 0$ such that for any radial Schwartz function f we have

$$f(x) = \sum_{n \geq n_0} a_n(x) f(\sqrt{2n}) + \sum_{n \geq n_0} b_n(x) f'(\sqrt{2n}) + \sum_{n \geq n_0} \widehat{a_n}(x) \widehat{f}(\sqrt{2n}) + \sum_{n \geq n_0} \widehat{b_n}(x) \widehat{f}'(\sqrt{2n})$$

Here $n_0 = 1$ for $d = 8$ and $n_0 = 2$ for $d = 24$.

This formula has been used to prove the following result.

Theorem (Cohn-Kumar-Miller-R.-Viazovska, 2019)

The E_8 and Leech lattices are universally optimal.

Uniqueness pairs with $A = \{\pm cn^\alpha\}$, $B = \{\pm dn^\beta\}$

Recent progress on uniqueness pairs of the form $A = \{\pm cn^\alpha\}$, $B = \{\pm dn^\beta\}$:

- Ramos and Sousa (2019) have shown that if α and β belong to a certain region $\Omega \subset \{\alpha, \beta \geq 0, \alpha + \beta < 1\}$, then (A, B) is a uniqueness pair
- Nazarov and Sodin have recently (May, 2020) showed that

$$\begin{aligned} \alpha + \beta < 1 \quad &\Rightarrow \quad (A, B) \text{ is a uniqueness pair,} \\ \alpha + \beta > 1 \quad &\Rightarrow \quad (A, B) \text{ is not a uniqueness pair.} \end{aligned}$$

  Moreover, for $\alpha + \beta = 1$ they have showed that (A, B) is a uniqueness pair if $2cd < \alpha^{-\alpha}\beta^{-\beta}$, and it is not a uniqueness pair if $2cd > \alpha^{-\alpha}\beta^{-\beta}$.

Conjecture

If $2cd = \alpha^{-\alpha}\beta^{-\beta}$, then $A = \{\pm cn^\alpha\}_{n\geq 0}$, $B = \{\pm dn^\beta\}_{n\geq 0}$ is a uniqueness pair.

Uniqueness pair with zeros of $\zeta(s)$

Finally, one can interpret the interpolation formula

$$f(z) = \sum_{n=1}^{\infty} \widehat{f}\left(\frac{\log n}{4\pi}\right) U_n(z) + \lim_{k\to\infty} \sum_{0<\gamma\le T_k} \sum_{j=0}^{m_\rho - 1} f^{(j)}\left(\frac{\rho - 1/2}{i}\right) V_{\rho,j}(z) \qquad (*)$$

as showing that $A = \{\frac{\rho-1/2}{i}\}_{\rho:\zeta(\rho)=0}$, $B = \{\pm\frac{\log n}{4\pi}\}$ is a uniqueness pair for an appropriate space of functions.

A caveat: $A \subset \mathbb{R}$ is equivalent to RH, and to get rid of derivatives, one needs $m_\rho = 1$

Let us now outline the proof of (*).

Hamburger's theorem

"Riemann's zeta function is uniquely determined by its functional equation"

Theorem (Hamburger)

If $\varphi\colon \mathbb{C} \to \mathbb{C}$ is a meromorhic function such that

(i) *for some polynomial P, the function $P(s)\varphi(s)$ is entire of finite genus*

(ii) $\Gamma_{\mathbb{R}}(s)\varphi(s) = \Gamma_{\mathbb{R}}(1-s)\varphi(1-s)$

(iii) $\varphi(s) = \sum_{n\geq 1} a_n n^{-s}$ *and the series converges in at least one point*

Then $\varphi(s)$ is a multiple of $\zeta(s)$.

Hecke's version of Hamburger's theorem

(iii)' $\varphi(s) = \sum_{n\geq 1} a_n n^{-s/2}$, but only a simple pole at $s = 1$ is allowed

Proof sketch.

Applying inverse Mellin transform to $\Gamma_{\mathbb{R}}(s)\varphi(s)$ gives $f(t) = \sum_{n\geq 1} a_n e^{-\pi n t}$ with

$$f(t) - t^{-1/2} f(1/t) = c - ct^{-1/2}\,.$$

Then $g(\tau) = f(\tau/i) - c$ is a holomorphic function on the upper half-plane $\mathbb{H}$ satisfying

$$g(\tau+2) - g(\tau) = 0, \qquad g(\tau) - (\tau/i)^{-1/2} g(-1/\tau) = 0\,.$$

Thus g is a modular form of weight $1/2$ for the theta group, and one can show that this forces $g(\tau) = -c\theta(\tau)$, where $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ is the Jacobi theta function. □

Dirichlet series and summation formulae

Let $L_1(s) = \sum_n \frac{a_n}{\lambda_n^s}$ and $L_2(s) = \sum_n \frac{b_n}{\mu_n^s}$ be two general Dirichlet series such that:

- they extend meromorphically to $\mathbb{C}$ with simple poles in the strip $0 \leq \mathrm{Re}(z) \leq k$
- they grow at most polynomially in vertical strips and satisfy

$$L_1(k - s) = L_2(s)$$

Proposition

For all nice f satisfying $f(k - s) = f(s)$ we have

$$\sum_{\rho} f(\rho)\, \mathrm{res}_{s=\rho} L_1(s) = \sum_{n \geq 0} a_n \mathcal{M}^{-1} f(\lambda_n) - \sum_{n \geq 0} b_n \mathcal{M}^{-1} f(\mu_n)\,.$$

Here $\mathcal{M}^{-1}(f)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) x^{-s} ds$ is the inverse Mellin transform.

Dirichlet series with a prescribed pole

Our result implies the existence of

$$\mathcal{A}_s(w) = \sum_{n \geq 1} \frac{\alpha_n(s)}{n^{w/2}}$$

such that

$$\Gamma_{\mathbb{R}}(w)\mathcal{A}_s(w) = -\Gamma_{\mathbb{R}}(1-w)\mathcal{A}_s(1-w)$$

and $\Gamma_{\mathbb{R}}(w)\mathcal{A}_s(w)$ has poles only at $0, 1$ and $s, 1-s$.

Then the summation formula corresponding to Dirichlet series

$$\frac{\mathcal{A}_s(w)}{\zeta(w)} = -\frac{\mathcal{A}_s(1-w)}{\zeta(1-w)}$$

gives our claim for all sufficiently nice functions f.

To get the claim for $f \in \mathcal{H}_1$ one needs good estimates for various sums involving $\alpha_n(s)$.

Explicit formulas for the basis functions

In terms of $\alpha_n(s)$ the basis functions U_n are given by

$$U_n\Big(\frac{s-1/2}{i}\Big) = \frac{1}{2}\zeta^*(s)\sum_{d^2|n}\mu(d)\alpha_{n/d^2}(s)$$

where $\zeta^*(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$.

Assuming that $m_\rho = 1$ we also have

$$V_{\rho,0}\Big(\frac{s-1/2}{i}\Big) = C_\rho\zeta^*(s)\mathcal{A}_s(\rho)\,, \qquad\qquad C_\rho = \frac{\Gamma_{\mathbb{R}}(\rho)}{(\zeta^*)'(\rho)}$$

Modular integrals for the theta group

Applying inverse Mellin transform to $\mathcal{A}_s(w)$ we get an equivalent problem: construct

$$F_s(\tau) = \sum_{n\geq 0} \alpha_n(s) e^{\pi i n \tau}\,, \qquad \operatorname{Im}\tau > 0$$

such that

$$F_s(\tau) + (\tau/i)^{-1/2} F_s(-1/\tau) = (\tau/i)^{-s/2} + (\tau/i)^{-(1-s)/2}\,.$$

In other words we are looking for a holomorphic function $F_s \colon \mathbb{H} \to \mathbb{C}$ satisfying

$$\begin{cases} F_s(\tau+2) - F_s(\tau) & = 0 \\ F_s(\tau) + (\tau/i)^{-1/2} F_s(-1/\tau) & = (\tau/i)^{-s/2} + (\tau/i)^{-(1-s)/2} \end{cases}$$

that does not grow quickly near the boundary of $\mathbb{H}$.

Modular integrals for the theta group

Solutions to the homogeneous system

$$\begin{cases} f(\tau+2) - f(\tau) & = 0 \\ f(\tau) + (\tau/i)^{-1/2} f(-1/\tau) & = 0 \end{cases}$$

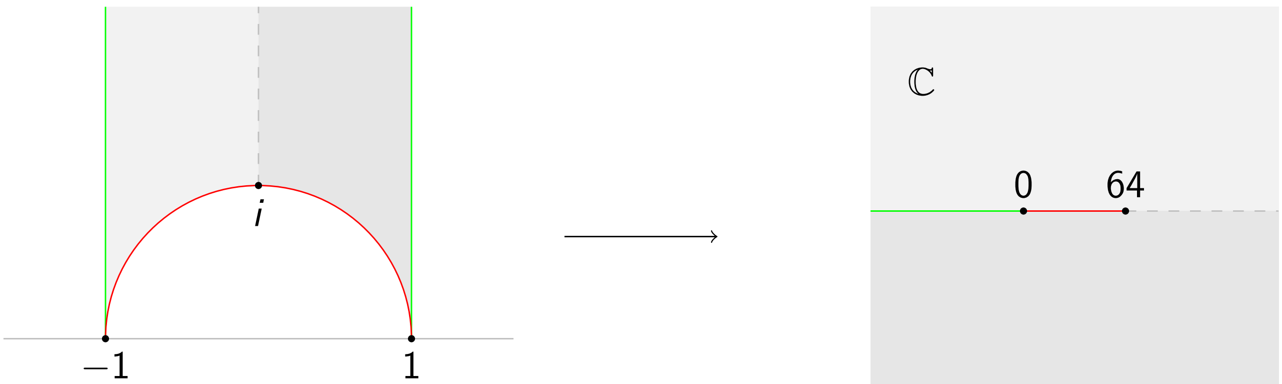
are (under suitable growth conditions) modular forms of weight $1/2$ for the group Γ_θ generated by $\tau \mapsto \tau + 2$ and $\tau \mapsto -1/\tau$.

The solution $F_s(\tau)$ to the inhomogeneous system can be written down explicitly (the idea comes from a paper by Duke, Imamoglu, and Toth on cycle integrals of the j-invariant): for τ in the fundamental domain one defines

$$F_s(\tau) = \frac{1}{2} \int_{-1}^{1} K(\tau, z)(z/i)^{-s/2} dz$$

where $K(\tau, z)$ is a Green-type kernel for the group Γ_θ. For general $\operatorname{Im} \tau > 0$ we can continue $F_s(\tau)$ analytically by iterating the functional equations.

Reformulation as a boundary value problem



The functional equation

$$F_s(\tau) + (\tau/i)^{-1/2} F_s(-1/\tau) = (\tau/i)^{-s/2} + (\tau/i)^{-(1-s)/2}$$

can be rewritten as a jump condition for a holomorphic function on $\mathbb{C} \setminus [0, 64]$ by applying a conformal map as above.

The contour integral for $F_s(\tau)$ then follows from Sokhotski-Plemelj formula.

Concluding remarks

The proof is relatively flexible and applies to many other L-functions.

- $L(\chi, s)$ for Dirichlet characters χ, with $\pm\frac{\log n}{4\pi}$ replaced by $\pm\frac{\log(qn)}{4\pi}$
- $\zeta_K(s)$ for imaginary quadratic fields, with $\pm\frac{\log n}{4\pi}$ replaced by $\pm\frac{\log(2D^{1/2}n)}{2\pi}$
- $L(f, s)$ for a cusp form f of weight k and level N, $\pm\frac{\log n}{4\pi} \mapsto \pm\frac{\log(2N^{1/2}n)}{2\pi}$

For L-functions whose functional equation involves a product of more than 2 gamma factors our approach does not give tight uniqueness pairs.

Since we use only the functional equation, we also get interpolation formulas from Dirichlet series that do not satisfy RH.

THANK YOU!