Littlewood-Richsrdson cone for symmetrizable Kac-Moody algebras

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra with the standard Cartan subalgebra $\mathfrak{h}$. Let $P_+ \subset \mathfrak{h}^*$ be the set of dominant integral weights. For $\lambda \in P_+$, let $L(\lambda)$ be the integrable, highest weight (irreducible) representation of $\mathfrak{g}$ with highest weight λ. For a positive integer s, define the *saturated tensor semigroup* as

$$\Gamma_s := (\lambda_1, \ldots, \lambda_s, \mu) \in P_+^{s+1} : \exists\, N > 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s).$$

Let $\Gamma_s(\mathbb{Q})$ be the rational cone generated by Γ_s. If $\mathfrak{g}$ is finite dimensional, $\Gamma_s(\mathbb{Q})$ is a polyhedral convex cone. Significant results were obtained for $\Gamma_s(\mathbb{Q})$ starting with the work of Weyl (1912), Horn (1962), Klyachko (1998), Knutson-Tao (1999), Berenstein-Sjamaar (2000), Belkale (2001), Kapovich-Leeb-Millson (2009) culminating in the work of Belkale-Kumar (2006) and Ressayre (2010) which produced an irredundant set of inequalities (for any finite dimensional $\mathfrak{g}$). In general, $\Gamma_s(\mathbb{Q})$ is neither polyhedral nor closed. The aim of these talks is to study $\Gamma_s(\mathbb{Q})$ in the infinite dimensional symmetrizable Kac-Moody case. In these talks, following a work of Brown-Kumar, we will give a set of necessary linear inequalities satisfied by $\Gamma_s(\mathbb{Q})$. They further conjectured that this set of inequalities should be sufficient. Now, Ressayre proved that indeed this set of inequalities is suffcient when $\mathfrak{g}$ is untwisted affine. He also obtained an explicit saturation factor for the semigroup Γ_2.

Even though we will try to make these lectures fairly self-contained, but some basic knowledge of representation theory of Kac-Moody Lie algebras as well as Geometric Invariant Theory will be helpful.