Littlewood-Richardsdson cone for symmetrizable Kac-Moody algebras

Let \( \mathfrak{g} \) be a symmetrizable Kac-Moody Lie algebra with the standard Cartan subalgebra \( \mathfrak{h} \). Let \( P_+ \subset \mathfrak{h}^* \) be the set of dominant integral weights. For \( \lambda \in P_+ \), let \( L(\lambda) \) be the integrable, highest weight (irreducible) representation of \( \mathfrak{g} \) with highest weight \( \lambda \). For a positive integer \( s \), define the saturated tensor semigroup as

\[
\Gamma_s := (\lambda_1, \ldots, \lambda_s, \mu) \in P_+^{s+1} : \exists N > 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s).
\]

Let \( \Gamma_s(\mathbb{Q}) \) be the rational cone generated by \( \Gamma_s \). If \( \mathfrak{g} \) is finite dimensional, \( \Gamma_s(\mathbb{Q}) \) is a polyhedral convex cone. Significant results were obtained for \( \Gamma_s(\mathbb{Q}) \) starting with the work of Weyl (1912), Horn (1962), Klyachko (1998), Knutson-Tao (1999), Berenstein-Sjamaar (2000), Belkale (2001), Kapovich-Leeb-Millson (2009) culminating in the work of Belkale-Kumar (2006) and Ressayre (2010) which produced an irredundant set of inequalities (for any finite dimensional \( \mathfrak{g} \)). In general, \( \Gamma_s(\mathbb{Q}) \) is neither polyhedral nor closed. The aim of these talks is to study \( \Gamma_s(\mathbb{Q}) \) in the infinite dimensional symmetrizable Kac-Moody case. In these talks, following a work of Brown-Kumar, we will give a set of necessary linear inequalities satisfied by \( \Gamma_s(\mathbb{Q}) \). They further conjectured that this set of inequalities should be sufficient. Now, Ressayre proved that indeed this set of inequalities is sufficient when \( \mathfrak{g} \) is untwisted affine. He also obtained an explicit saturation factor for the semigroup \( \Gamma_2 \).

Even though we will try to make these lectures fairly self-contained, but some basic knowledge of representation theory of Kac-Moody Lie algebras as well as Geometric Invariant Theory will be helpful.