Hermitian Eigenvalue Problem and its Generalization to Arbitrary Reductive Groups: A Survey

Shrawan Kumar

Hermitian eigenvalue problem

For any $n \times n$ Hermitian matrix A, let $\lambda_A = (\lambda_1 \geq \cdots \geq \lambda_n)$ be its set of eigenvalues written in descending order. (Recall that all the eigenvalues of a Hermitian matrix are real.) We recall the following classical problem.

Problem 1

(The Hermitian eigenvalue problem) Given two n-tuples of nonincreasing real numbers: $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_n)$ *and* $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_n)$, *determine all possible* $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_n)$ *such that there exist Hermitian matrices* A, B, C *with* $\lambda_A = \boldsymbol{\lambda}, \lambda_B = \boldsymbol{\mu}, \lambda_C = \boldsymbol{\nu}$ *and* $C = A + B$.

Said imprecisely, the problem asks the possible eigenvalues of the sum of two Hermitian matrices with fixed eigenvalues.

A conjectural solution of the above problem was given by Horn in 1962.

For any positive integer $r < n$, inductively define the set S_r^n as the set of triples (I, J, K) of subsets of $[n] := \{1, \ldots, n\}$ of cardinality r such that

$$\sum_{i \in I} i + \sum_{j \in J} j = r(r+1)/2 + \sum_{k \in K} k \tag{1}$$

and for all $0 < p < r$ and $(F, G, H) \in S_p^r$ the following inequality holds:

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq p(p+1)/2 + \sum_{h \in H} k_h. \tag{2}$$

Now, Horn conjectured the following.

Conjecture 2

(Horn) A triple $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$ occurs as eigenvalues of Hermitian $n \times n$ matrices A, B, C respectively such that $C = A + B$ if and only if

$$\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i,$$

and for all $1 \leq r < n$ and all triples $(I, J, K) \in S_r^n$, we have

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j.$$

Of course, the first identity is nothing but the trace identity.

Remark. Even though this problem goes back to the nineteenth century, the first significant result was given by H. Weyl in 1912:

$$\nu_{i+j-1} \leq \lambda_i + \mu_j \text{ whenever } i + j - 1 \leq n.$$

Exercise. Show that for $n = 2$, the trace identity and the Weyl inequality are sufficient.
K. Fan found some other inequalities in 1949 followed by Lidskii (1950). The full set of inequalities given above are due to Horn. Horn's above conjecture was settled in the affirmative by combining the work of Klyachko (1998) with the work of Knutson-Tao (1999) on the 'saturation' problem. The above system of inequalities is overdetermined. Belkale (2001) came up with a subset of the above set of inequalities which forms an irredundant system of inequalities as proved by Knutson-Tao-Woodward (2004).

Generalization of the eigenvalue problem

Now we will discuss a generalization of the above Hermitian eigenvalue problem to an arbitrary complex semisimple group. (A further generalization to any reductive group follows fairly easily from the semisimple case.)

So, let G be a connected, simply-connected, semisimple complex algebraic group. We fix a Borel subgroup B, a maximal torus H, and a maximal compact subgroup K. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{k}$ respectively. Let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$) and let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R^+$ be the set of simple roots. There is a natural homeomorphism $\delta : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation and $\mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_j(h) \geq 0\}$ (for all the simple roots α_j) is the positive Weyl chamber in $\mathfrak{h}$. The inverse map δ^{-1} takes any $h \in \mathfrak{h}_+$ to the K-conjugacy class of ih.

For any positive integer s, define the set $\Gamma(s) :=$

$$\{(h_1, \ldots, h_s) \in \mathfrak{h}_+^s \mid \exists (k_1, \ldots, k_s) \in \mathfrak{k}^s :$$

$$\sum_{j=1}^{s} k_j = 0 \text{ and } \delta(k_j) = h_j \forall j = 1, \ldots, s\}.$$

Following is the generalization of the Hermitian eigenvalue problem to an arbitrary G. (The case $G = Gl(n)$ and $s = 3$ specializes to the problem discussed in the beginning if we replace C by $-C$.)

Problem 3

Describe the set $\Gamma(s)$*.*

By virtue of the convexity result in symplectic geometry, the subset $\Gamma(s) \subset \mathfrak{h}_+^s$ is a convex polyhedral cone (defined by certain inequalities). The aim is to find these inequalities describing $\Gamma(s)$ explicitly.

Before we can give a solution of the problem, we need some more notation.

Let $P \supset B$ be a standard parabolic subgroup with Lie algebra $\mathfrak{p}$ and let $\mathfrak{l}$ be its unique Levi component containing the Cartan subalgebra $\mathfrak{h}$. Let $\Delta(P) \subset \Delta$ be the set of simple roots contained in the set of roots of $\mathfrak{l}$. For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\ 1 \leq i \leq \ell. \tag{3}$$

Let W_P be the Weyl group of P (which is, by definition, the Weyl Group of the Levi component L), then in each coset of W/W_P we have a unique member w of minimal length. Let W^P be the set of the minimal length representatives in the cosets of W/W_P.
For any $w \in W^P$, define the (shifted) Schubert cell:

$$\Lambda_w^P := w^{-1}BwP \subset G/P.$$

Then, it is a locally closed subvariety of G/P isomorphic with the affine space $\mathbb{A}^{\ell(w)}, \ell(w)$ being the length of w . Its closure is denoted by $\bar{\Lambda}_w^P$, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. Let $\mu(\bar{\Lambda}_w^P)$ denote the fundamental class of $\bar{\Lambda}_w^P$ considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P. Then, from the Bruhat decomposition, the elements $\{\mu(\bar{\Lambda}_w^P)\}_{w \in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P, \mathbb{Z})$. Let $\{\epsilon_w^P\}_{w \in W^P}$ be the dual basis of the singular cohomology with integral coefficients $H^*(G/P, \mathbb{Z})$, i.e., for any $v, w \in W^P$ we have

$$\epsilon_v^P(\mu(\bar{\Lambda}_w^P)) = \delta_{v,w}.$$

Given a standard maximal parabolic subgroup P, let ω_P denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i^\vee) = 1$, if $\alpha_i \in \Delta \setminus \Delta(P)$ and 0 otherwise, where $\alpha_i^\vee$ is the fundamental coroot corresponding to the simple root α_i.

Deformation of Cup Product in $H^*(G/P)$

Let P be any standard parabolic subgroup of G. Write the standard cup product in $H^*(G/P,\mathbb{Z})$ in the $\{\epsilon_w^P\}$ basis as follows:

$$[\epsilon_u^P]\cdot[\epsilon_v^P] = \sum_{w\in W^P} d_{u,v}^w[\epsilon_w^P]. \tag{4}$$

Introduce the indeterminates τ_i for each $\alpha_i \in \Delta\setminus\Delta(P)$ and define a deformed cup product $\odot$ as follows:

$$\epsilon_u^P \odot \epsilon_v^P = \sum_{w\in W^P} \Big(\prod_{\alpha_i\in\Delta\setminus\Delta(P)} \tau_i^{(u^{-1}\rho+v^{-1}\rho-w^{-1}\rho-\rho)(x_i)}\Big) d_{u,v}^w \epsilon_w^P,$$

where ρ is the (usual) half sum of positive roots of $\mathfrak{g}$. This deformed product was introduced by Belkale-Kumar (2006). This product should not be confused with the small quantum cohomology product of G/P.

By using the Geometric Invariant Theory, one proves that whenever $d^w_{u,v}$ is nonzero, the exponent of τ_i in the above is a nonnegative integer. Moreover, the product $\odot$ is associative (and clearly commutative).

The cohomology algebra of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P,\mathbb{Z}) \otimes \mathbb{Z}[\tau_i], \odot)$ is denoted by $(H^*(G/P,\mathbb{Z}), \odot_0)$. Thus, as a $\mathbb{Z}$-module, this is the same as the singular cohomology $H^*(G/P,\mathbb{Z})$ and under the product $\odot_0$ it is associative (and commutative). Moreover, it continues to satisfy the Poincaré duality. Further, it can be proved that for a cominuscule maximal paparbolic P (i.e., the simple root $\alpha_P \in \Delta \setminus \Delta(P)$ occurs with coefficient one in the highest root θ of R^+), the product $\odot_0$ coincides with the standard cup product.

Now we are ready to state the main result on solution of the eigenvalue problem for any G stated above.

Theorem 4

(due to Belkale-Kumar, 2006) Let $(h_1, \ldots, h_s) \in \mathfrak{h}_+^s$*. Then, the following are equivalent:*
(a) $(h_1, \ldots, h_s) \in \Gamma(s)$.
(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ *such that*

$$\epsilon_{w_1}^P \odot_0 \cdots \odot_0 \epsilon_{w_s}^P = \epsilon_o^P \in \big(H^*(G/P, \mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$\omega_P(\sum_{j=1}^{s} w_j^{-1} h_j) \geq 0,$$

where ϵ_o^P *is the (top) fundamental class (which is the oriented integral generator of* $H^{\text{top}}(G/P, \mathbb{Z})$*).*

These set of inequalities are now generally referred to as the *Belkale-Kumar set of inequalities*.

The 'explicit' determination of $\Gamma(s)$ via the above Theorem hinges upon understanding the product $\odot_0$ in $H^*(G/P)$ in the Schubert basis, for all the maximal parabolic subgroups P. Clearly, the product $\odot_0$ is easier to understand than the usual cup product (which is the subject matter of *Schubert Calculus*) since in general 'many more' terms in the product $\odot_0$ in the Schubert basis drop out.
The cone $\Gamma(3)$ is quite explicitly determined for any simple G:

- of rank 2 by Kapovich-Leeb-Millson (2009)
- any simple G of rank 3 by Kumar-Leeb-Millson (2003), and
- for $G = \text{Spin}(8)$ by Kapovich-Kumar-Millson (2009).

It has $50, 102, 102, 306$ facets for G of type A_3, B_3, C_3, D_4 respectively.

Remark. The above theorem specializes to a solution of the Hermitian eigenvalue problem if we take $G = Gl(n)$. In this case, every maximal parabolic subgroup P is minuscule and hence, as mentioned earlier, the deformed product $\odot_0$ in $H^*(G/P)$ coincides with the standard cup product. In this case, the above theorem was obtained by Klyachko (1998) with a refinement by Belkale (2001). (The set of inequalities (b) for $G = Gl(n)$ in general is much smaller than the set of Horn inequalities discussed earlier. Further, as shown by Knutson-Tao-Woodward (2004), the set of inequalities (b) is an irredundant system for $G = Gl(n)$. In fact, as we will see later that it is true for any G.) If we replace the product $\odot_0$ in (b) by the standard cup product, then the equivalence of (a) and (b) for general G was proved by Kapovich-Leeb-Millson (2009) following an analogous slightly weaker result proved by Berenstein-Sjamaar (2000).

It may be mentioned that replacing the product $\odot_0$ in (b) by the standard cup product, we get far more inequalities for groups other than $Gl(n)$ (or $\mathrm{SL}(n)$). For example, as shown by Kumar-Leeb-Millson (2003), for G of type B_3 or C_3, the standard cup product gives rise to 135 inequalities, whereas the new product gives only 102 inequalities.

My interest in the eigenvalue problem stems from the problem of tensor product decomposition. Specifically, for any dominant integral weight $\lambda \in \mathfrak{h}^*$ (i.e., $\lambda(\alpha_i^\vee) \in \mathbb{Z}_+$ for each simple coroot $\alpha_i^\vee$), let $V(\lambda)$ be the finite dimensional irreducible G-module with highest weight λ. *Given dominant integral weights* $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$, *a classical and a very central problem is to determine which irreducible representations* $V(\nu)$ *occur in the tensor product* $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$*?* By taking the tensor product of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$ with the dual representation $V(\nu)^*$, (and replacing s by $s+1$) we can reformulate the above question more symmetrically as follows.

Problem 5

Determine the set of s-tuples $(\lambda_1, \dots, \lambda_s)$ *of dominant integral weights such that the tensor product* $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$ *has a nonzero G-invariant subspace.*

This problem in general seems quite hard. So, let us pose the following weaker *saturated tensor product* problem.

Problem 6

Determine the set $\hat{\Gamma}(s)$ *of s-tuples* $(\lambda_1, \ldots, \lambda_s)$ *of dominant rational weights such that the tensor product* $V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)$ *has a nonzero G-invariant subspace for some positive integer N, where we call a weight a dominant rational weight if its some positive integral multiple is a dominant integral weight.*

The above saturated tensor product problem is parallel to the eigenvalue problem because of the following result. Let $D := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{R}_+ \forall i\}$ be the set of dominant real weights. Then, under the Killing form, we have an identification $\mathfrak{h}_+ \to D$. Under this identification, x_i corresponds with $2\omega_i/\langle\alpha_i, \alpha_i\rangle$, where ω_i is the i-th fundamental weight.

The following result due to Sjamaar (1998) (proved by using the moment map in symplectic geometry) shows that the eigenvalue problem coincides with the saturated tensor product problem.

Proposition 7

Under the identification of $\mathfrak{h}_+$ with D (and hence of $\mathfrak{h}_+^s$ with D^s), $\Gamma(s)$ corresponds to the closure of $\hat{\Gamma}(s)$. In fact, $\hat{\Gamma}(s)$ consists of the rational points of the image of $\Gamma(s)$.

The following theorem is the main result on the saturated tensor product decomposition parallel to the solution of the eigenvalue problem.

Theorem 8

(due to Belkale-Kumar, 2006) Let $(\lambda_1, \ldots, \lambda_s)$ be a s-tuple of dominant integral weights. Then, the following are equivalent:

(i) *For some integer $N > 0$, the tensor product $V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)$ has a nonzero G-invariant subspace.*

(ii) *For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that*

$$\epsilon^P_{w_1} \odot_0 \cdots \odot_0 \epsilon^P_{w_s} = \epsilon^P_o \in \left(H^*(G/P, \mathbb{Z}), \odot_0\right),$$

the following inequality holds:

$$I^P_{(w_1,\ldots,w_s)}: \qquad \sum_{j=1}^{s} \lambda_j(w_j x_{i_P}) \geq 0, \tag{5}$$

where α_{i_P} is the simple root in $\Delta \setminus \Delta(P)$.

I have said nothing so far about the proofs, nor can I say much for lack of time. But let me mention that Theorem 4 on the eigenvalue problem for an arbitrary G follows from Theorem 8 and Proposition 7. The proof of Theorem 8 makes essential use of the Geometric Invariant Theory, specifically the Hilbert-Mumford criterion for semistability and Kempf's *maximally destabilizing one parameter subgroups* and Kempf's *parabolic subgroups* associated to unstable points. In addition, the notion of Levi-movability (defined below) plays a fundamental role in the proofs.
Also, the new product $\odot_o$ in the cohomology of the flag variety G/P is intimately connected with the Lie algebra cohomology of the nil-radical of the parabolic subalgebra $\mathfrak{p}$.

Here is the definition of Levi-movability: Let P be any standard parabolic subgroup of G with Levi component L. Let $w_1, \dots, w_s \in W^P$ be such that

$$\sum_{j=1}^{s} \operatorname{codim} \Lambda_{w_j}^P = \dim G/P. \tag{6}$$

This of course is equivalent to the condition:

$$\sum_{j=1}^{s} \ell(w_j) = (s-1) \dim G/P. \tag{7}$$

Then, the s-tuple $(w_1, \dots, w_s)$ is called *Levi-movable* for short *L-movable* if, for generic $(l_1, \dots, l_s) \in L^s$, the intersection $l_1 \Lambda_{w_1} \cap \cdots \cap l_s \Lambda_{w_s}$ is transverse at e.

The following result due to Ressayre (2010) shows that the system of inequalities given by the above Theorem is an irredundant system for any G.

Theorem 9

The Belkale-Kumar set of inequalities provided by the (ii)-part of the above theorem is an irredundant system of inequalities describing the rational cone $\hat{\Gamma}(s)$ *inside* D^s*, i.e., the hyperplanes given by the equality in* $I^P_{(w_1,\dots,w_s)}$ *are precisely those facets of the cone* $\hat{\Gamma}(s)$ *which intersect the interior of* D^s*.*

Saturation Problem

The *saturation problem* aims at connecting the tensor product semigroup $T(s) = T_G(s)$ (consisting of the s-tuples $(\lambda_1, \dots, \lambda_s)$ of dominant integral weights such that the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$ has a nonzero G-invariant) with the saturated tensor product semigroup $\hat{\Gamma}(s)_{\mathbb{Z}}$ (consisting of the s-tuples $(\lambda_1, \dots, \lambda_s)$ of dominant integral weights such that the tensor product $V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)$ has a nonzero G-invariant for some $N > 0$).

We begin with the following definition. We take $s = 3$ as this is the most relevant case to the tensor product decomposition.

Definition 10

An integer $d \geq 1$ is called a *saturation factor* for G, if for any $(\lambda, \mu, \nu) \in \hat{\Gamma}(3)_{\mathbb{Z}}$ such that $\lambda + \mu + \nu \in Q$, then $(d\lambda, d\mu, d\nu) \in T(3)$, where Q is the root lattice of G.
Of course, if d is a saturation factor then so is its any multiple.

If $d = 1$ is a saturation factor for G, we say that the *saturation property holds for G*.

The *saturation theorem* of Knutson-Tao (1999) mentioned earlier, proved by using their 'honeycomb model' asserts the following. Other proofs of their result are given by Derksen-Weyman (2000), Belkale (2006) and Kapovich-Millson (2008).

Theorem 11

The saturation property holds for $G = \mathrm{SL}(n)$.

The following general result (though not optimal) on saturation factor is obtained by Kapovich-Millson by using the geometry of geodesics in Euclidean buildings and Littelmann's path model.

Theorem 12

(Kapovich-Millson, 2008) For any connected simple G, $d = k_{\mathfrak{g}}^2$ *is a saturated factor, where* $k_{\mathfrak{g}}$ *is the least common multiple of the coefficients of the highest root* θ *of the Lie algebra* $\mathfrak{g}$ *of G written in terms of the simple roots* $\{\alpha_1, \ldots, \alpha_\ell\}$.

Observe that the value of $k_{\mathfrak{g}}$ *is* 1 *for* $\mathfrak{g}$ *of type* $A_\ell(\ell \geq 1)$*; it is* 2 *for* $\mathfrak{g}$ *of type* $B_\ell(\ell \geq 2), C_\ell(\ell \geq 3), D_\ell(\ell \geq 4)$*; and it is 6, 12, 60, 12, 6 for* $\mathfrak{g}$ *of type* E_6, E_7, E_8, F_4, G_2 *respectively.*

Kapovich-Millson determined $T_G(3)$ explicitly for $G = \mathrm{Sp}(4)$ and G_2 (2006). In particular, from their description, the following theorem follows easily.

Theorem 13

The saturation property does not hold for either $G = \mathrm{Sp}(4)$ *or* G_2*. Moreover,* 2 *is a saturation factor (and no odd integer* d *is a saturation factor) for* $\mathrm{Sp}(4)$*, whereas both of* $2, 3$ *are saturation factors for* G_2 *(and hence any integer* $d > 1$ *is a saturation factor for* G_2*).*

Kapovich-Millson made the following very interesting conjecture (2006):

Conjecture 14

If G is simply-laced, then the saturation property holds for G.

Apart from $G = \mathrm{SL}(n)$, the only other simply-connected, simple, simply-laced group G for which the above conjecture is known so far is $G = \mathrm{Spin}(8)$, proved by Kapovich-Kumar-Millson (2009) by explicit calculation using the Belkale-Kumar set of inequalities.

Finally, we have the following improvement of the general saturatuion Theorem due to Kapovich-Millson applied to the groups $\mathrm{SO}(2\ell+1)$ and $\mathrm{Sp}(2\ell)$.

Theorem 15

(Belkale-Kumar, 2010) For the groups $\mathrm{SO}(2\ell+1)$ *and* $\mathrm{Sp}(2\ell)$*, 2 is a saturation factor.*

(Observe that the general result of Kapovich-Millson gives a saturation factor of 4 in these cases.)
The proof of the above theorem relies on the following theorem due to Belkale-Kumar (2010).

Theorem 16

Let $(\lambda^1, \dots, \lambda^s) \in T_{\mathrm{SL}(2\ell)}(s)$*. Then,* $(\lambda^1_C, \dots, \lambda^s_C) \in T_{(\mathrm{Sp}(2\ell))}(s)$*, where* λ^j_C *is the restriction of* λ^j *to the maximal torus of* $\mathrm{Sp}(2\ell)$*. A similar result is true for* $\mathrm{Sp}(2\ell)$ *replaced by* $\mathrm{SO}(2\ell+1)$*.*

Belkale-Kumar made the following conjecture:

Let G be a simply-connected, simple complex algebraic group and let σ be a diagram automorphism of G with fixed subgroup K.

Conjecture 17

Let $(\lambda^1, \ldots, \lambda^s) \in T_G(s)$. Then, $(\lambda_K^1, \ldots, \lambda_K^s) \in T_K(s)$, where λ_K^j is the restriction of λ^j to the maximal torus of K.
(Observe that λ_K is dominant for K for any dominant character λ for G with respect to the Borel subgroup $B^K := B^\sigma$ of K.)

We also mention the following 'rigidity' result (conjectured by Fulton) due to Knutson-Tao-Woodward (2004) proved by combinatorial methods. There are now geometric poofs of the theorem by Belkale (2007) and Ressayre (2009).

Theorem 18

Let $G = \mathrm{SL}(n)$ *and let* $\lambda, \mu, \nu \in \Lambda^+$. *If* $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^G$ *is one-dimensional then so is* $[V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^G$*, for any* $N \geq 1$.

The direct generalization of this theorem for other groups is, in general, false. But, a certain cohomological reinterpretation of the theorem remains true for any G as proved by Belkale-Kumar-Ressayre (2010).

Remark 19

We mention the classical Littlewood-Richardson theorem for the tensor product decomposition of irreducible polynomial representations of $\mathrm{GL}(n)$ and its generalization by Littelmann for any G via his *LS path model*.

In addition, we mention that Berenstein-Zelevinsky determined the tensor product multiplicities as the number of lattice points in some convex polytope.

For the tensor product multiplicities, there is an approach by Lusztig via his *canonical bases*. Similarly, there is an approach by Kashiwara via his *crystal bases*. For lack of time, we do not give the details.

Final Remark. I do not have time to talk about the multiplicative analogue of the eigenvalue problem.

DANKE SCHÖN!