Counter-examples to the Hirsch conjecture

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http://personales.unican.es/santosf/Hirsch

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Polyhedra and polytopes

Definition

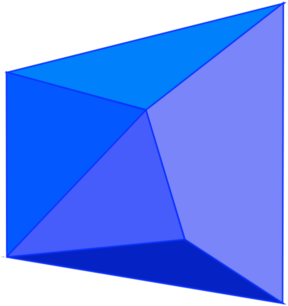
A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.

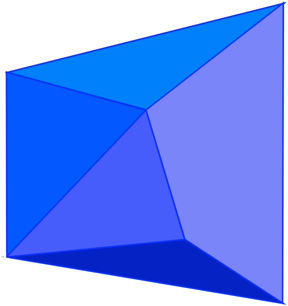


The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, but not conversely.



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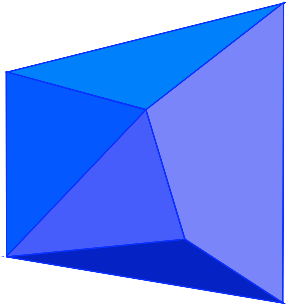

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Faces of P

Let P be a polytope (or polyhedron) and let

$$H = \{x \in \mathbb{R}^d : a_1x_1 + \cdots a_dx_d \leq a_0\}$$

be an affine half-space.

If $P \subset H$ we say that $\partial H \cap P$ is a face of P.

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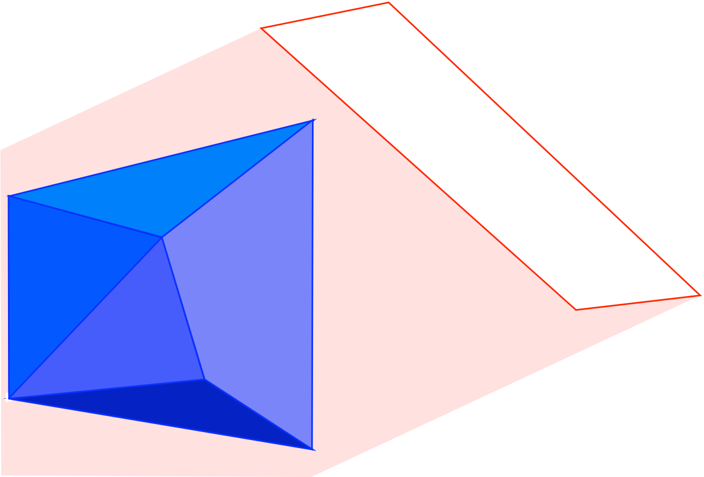
Faces of P

The “empty face” of P.



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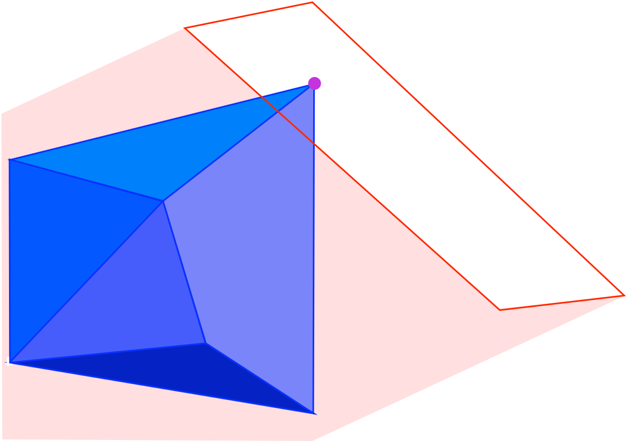
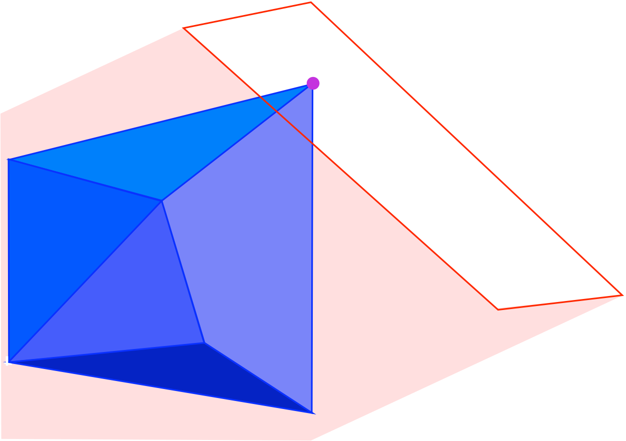


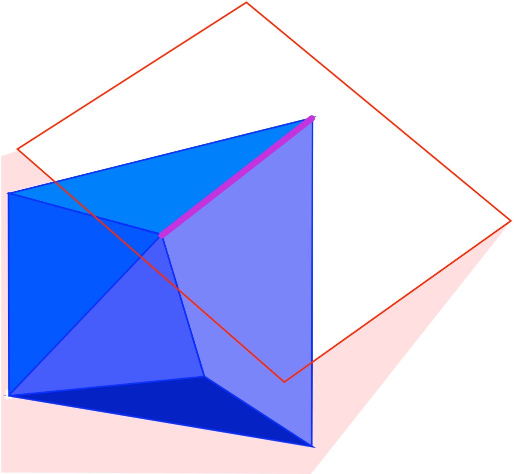
Faces of P

Faces of dimension 0 are called vertices.



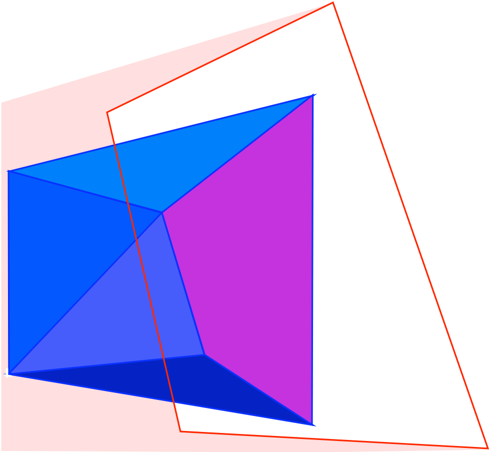
Faces of P

Faces of dimension 1 are called edges.



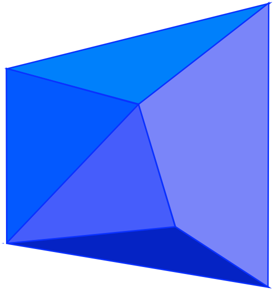
Faces of P

Faces of dimension $d-1$ (codimension 1) are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a (finite, undirected) graph.



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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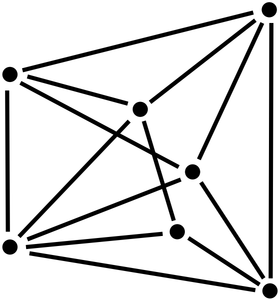

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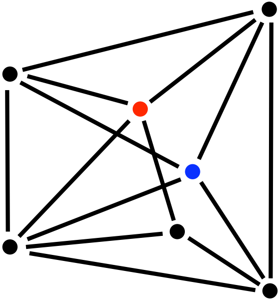

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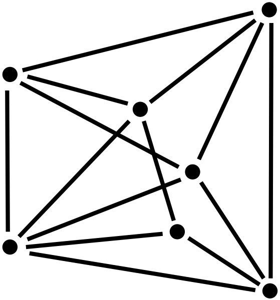

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) := \max\{d(u,v) : u,v \in V\}.$$

The Hirsch conjecture

Let $\delta(P)$ denote the diameter of the graph of a polytope P.

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

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Fifty three years later...

Theorem (S. 2010+)

There is a 43-dim. polytope with 86 facets and diameter ≥ 44.

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There is a 20*-dim. polytope with* 40 *facets and diameter* $\geq$ 21*.*

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Corollary

There is an infinite family of non-Hirsch polytopes with diameter $\sim (1 + \epsilon)n$*, even in fixed dimension. (Best so far:* $\epsilon = 1/20$*).*

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Remark

To this day, we do not know any polynomial upper bound for $\delta(P)$, in terms of n and d (polynomial Hirsch Conjecture)

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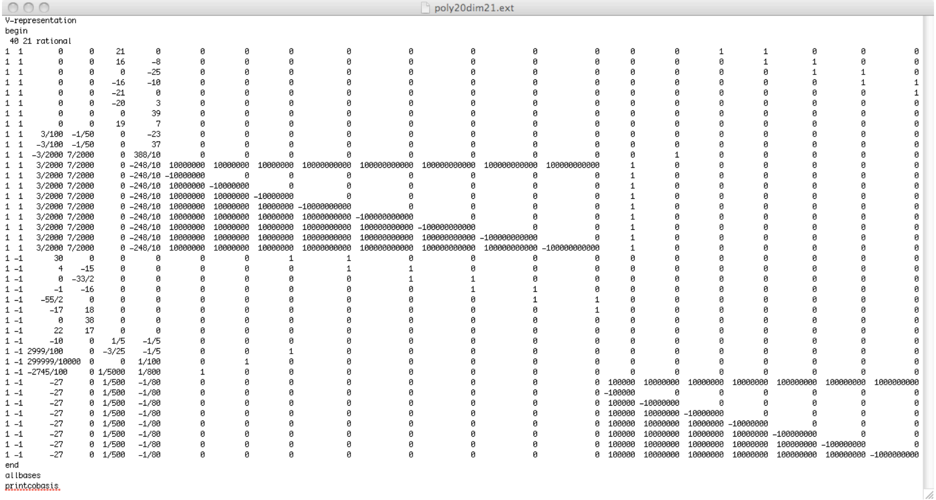
This polytope has been explicitly computed. It has 36, 442 vertices, and diameter 21.



```
V-representation
begin
 40 21 rational
1  1        0      0     21       0         0         0         0            0             0             0             0             0       0         0         1         1          0          0           0
1  1        0      0     16      -8         0         0         0            0             0             0             0             0       0         0         0         1          1          0           0
1  1        0      0      0     -25         0         0         0            0             0             0             0             0       0         0         0         0          1          1           0
1  1        0      0    -16     -10         0         0         0            0             0             0             0             0       0         0         0         0          0          1           1
1  1        0      0    -21       0         0         0         0            0             0             0             0             0       0         0         0         0          0          0           1
1  1        0      0    -20       3         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1  1        0      0      0      39         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1  1        0      0     19       7         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1  1    3/100  -1/50      0     -23         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1  1   -3/100  -1/50      0      37         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1  1  -3/2000 7/2000      0  388/10         0         0         0            0             0             0             0             0       0         1         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000  100000000000  100000000000       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10 -10000000         0         0            0             0             0             0             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000 -10000000         0            0             0             0             0             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000 -10000000            0             0             0             0             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000  10000000 -10000000000             0             0             0             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000  10000000  10000000000 -100000000000             0             0             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000  10000000  10000000000  100000000000 -100000000000             0             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000 -100000000000             0       1         0         0         0          0          0           0
1  1   3/2000 7/2000      0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000  100000000000 -100000000000       1         0         0         0          0          0           0
1 -1       30      0      0       0         0         0         1            1             0             0             0             0       0         0         0         0          0          0           0
1 -1        4    -15      0       0         0         0         0            1             1             0             0             0       0         0         0         0          0          0           0
1 -1        0  -33/2      0       0         0         0         0            0             1             1             0             0       0         0         0         0          0          0           0
1 -1       -1    -16      0       0         0         0         0            0             0             1             1             0       0         0         0         0          0          0           0
1 -1    -55/2      0      0       0         0         0         0            0             0             0             1             1       0         0         0         0          0          0           0
1 -1      -17     18      0       0         0         0         0            0             0             0             0             1       0         0         0         0          0          0           0
1 -1        0     38      0       0         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1 -1       22     17      0       0         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1 -1      -10      0    1/5    -1/5         0         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1 -1 2999/100      0  -3/25    -1/5         0         0         1            0             0             0             0             0       0         0         0         0          0          0           0
1 -1 299999/10000  0      0   1/100         0         1         0            0             0             0             0             0       0         0         0         0          0          0           0
1 -1 -2745/100     0 1/5000   1/800         1         0         0            0             0             0             0             0       0         0         0         0          0          0           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000  10000000  10000000  10000000  100000000  100000000  1000000000
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0 -100000         0         0         0          0          0           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000 -10000000         0         0          0          0           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000  10000000 -10000000         0          0          0           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000  10000000  10000000 -10000000          0          0           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000  10000000  10000000  10000000 -100000000          0           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000  10000000  10000000  10000000  100000000 -100000000           0
1 -1      -27      0  1/500   -1/80         0         0         0            0             0             0             0             0  100000  10000000  10000000  10000000  100000000  100000000 -1000000000
end
allbases
printcobasis
```

A **quasi**-polynomial bound, and a bound in fixed dimension

Theorem [Kalai-Kleitman 1992]

For every d-polytope with n facets:

$$\delta(P) \leq n^{\log_2 d+2}.$$

Theorem [Barnette 1967, Larman 1970]

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A linear program is the problem of maximization (or minimization) of a linear functional subject to linear inequality constraints. That is: finding $\max\{c \cdot x : x \in \mathbb{R}^d, Mx \leq b\}$ for given $c \in \mathbb{R}^d, b \in \mathbb{R}^n, M \in \mathbb{R}^{d \times n}$.

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> *"If one would take statistics about which* **mathematical problem** *is using up* **most of the computer time in the world***, then (not including database handling problems like sorting and searching) the answer would probably be linear programming."*
>
> (László Lovász, 1980)

Conection to the Hirsch conjecture

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets and d dimensions.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves linear programming by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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- For most of the pivot rules devised so far there is an analogue of the Klee-Minty cube, which makes the simplex method take an exponential number of steps:

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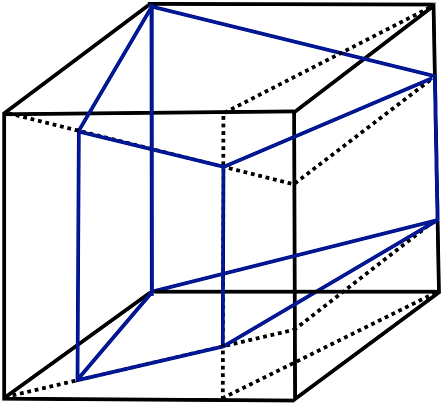

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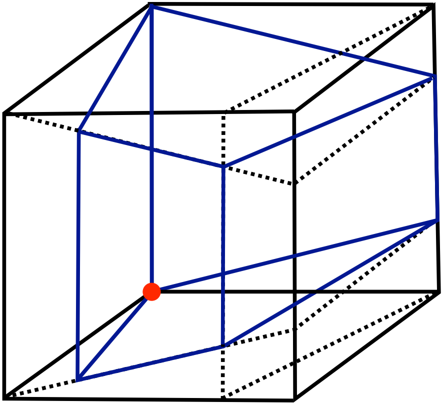

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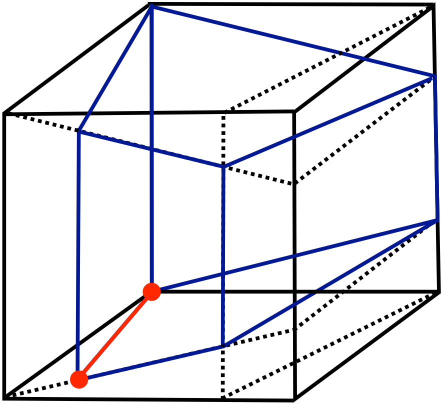

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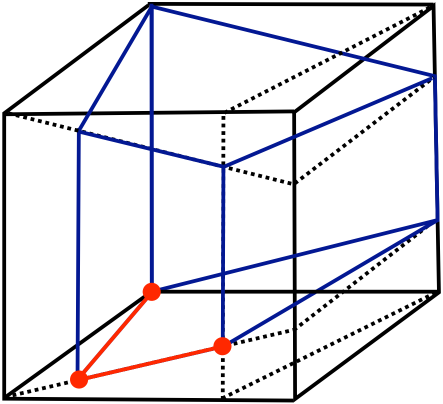

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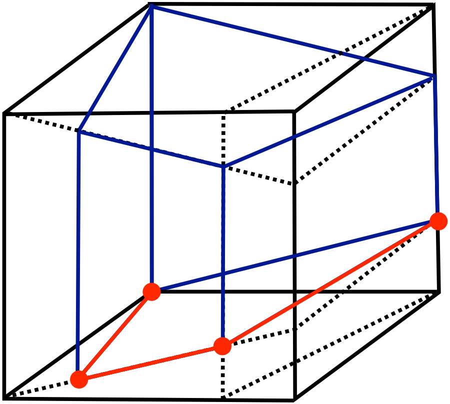

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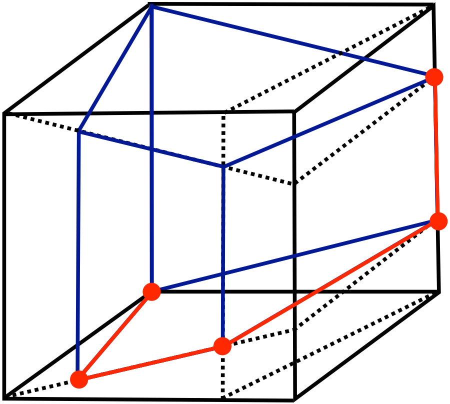

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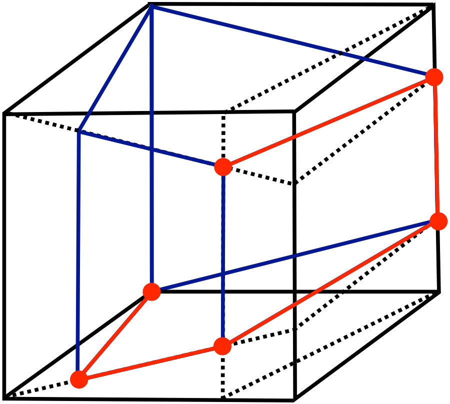

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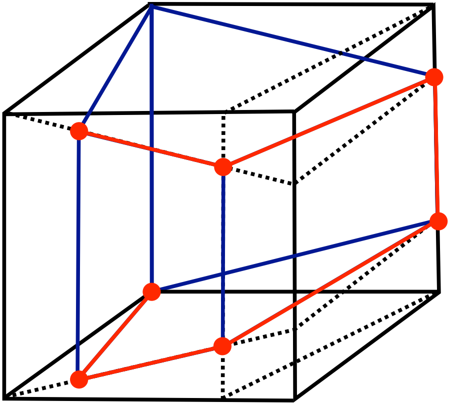

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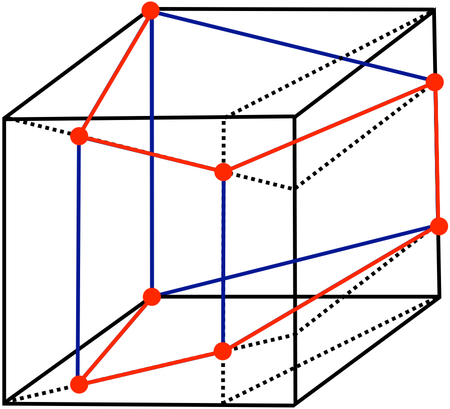

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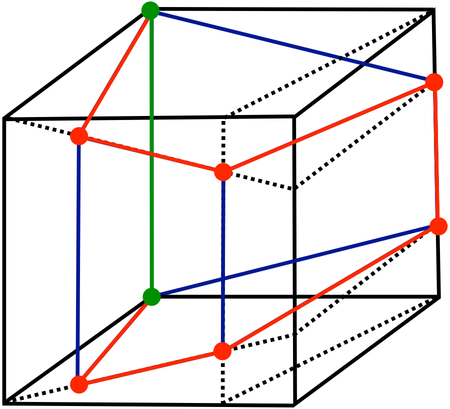

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Besides, the known polynomial algorithms for linear programming known are not *strongly polynomial*: They are polynomial in the bit model of complexity (Turing machine) but not in the arithmetic model (real RAM machine).

Finding strongly polynomial algorithms for linear programming is one of the "mathematical problems for the 21st century" according to [Smale 2000]. A polynomial pivot rule would solve this problem in the affirmative.

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- It holds with equality in simplices ($n=d+1$, $\delta=1$) and cubes ($n=2d$, $\delta=d$).
- It holds for all 0-1 polytopes [Naddef 1989] and for 3-polytopes [Klee 1966].
- If P and Q satisfy it, then so does $P\times Q$: $\delta(P\times Q)=\delta(P)+\delta(Q)$. In particular:

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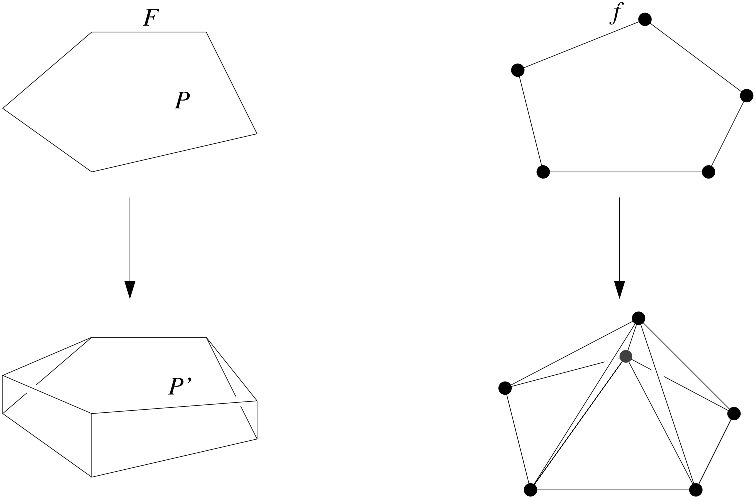

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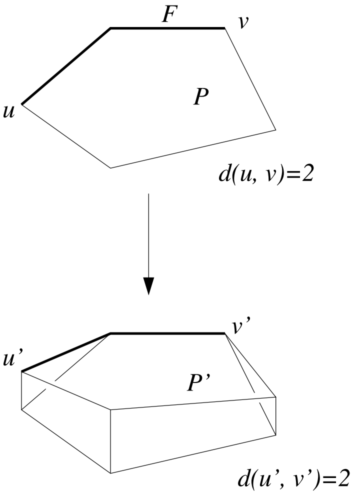

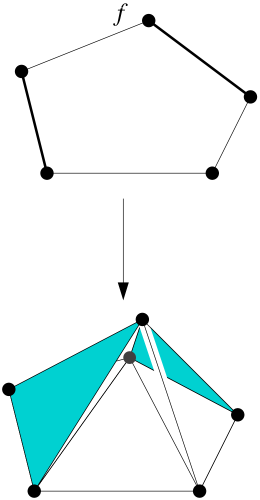

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Let P be a polytope of dimension d, with $n > 2d$ facets and diameter λ. Then there is another polytope P' of dimension $d + 1$, with $n + 1$ facets and diameter λ.

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Our construction of counter-examples has two ingredients:

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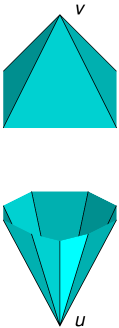
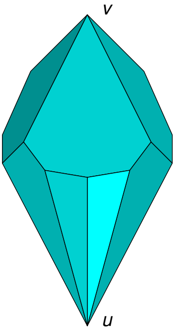
Spindles and prismatoids

Definition

A spindle is a polytope P with two distinguished vertices u and v such that every facet contains either u or v.





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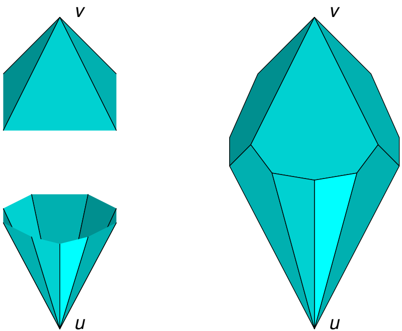

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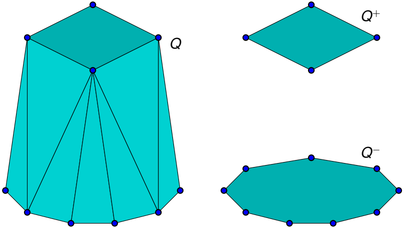

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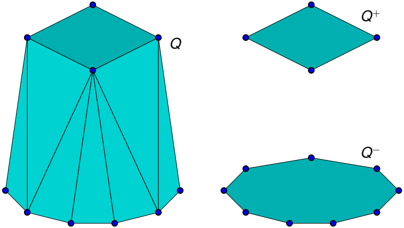

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Theorem (Strong *d*-step, spindle version)

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That is: we can increase the dimension, number of facets *and length* of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \delta + n - 2d > n-d$) that violates the Hirsch conjecture.

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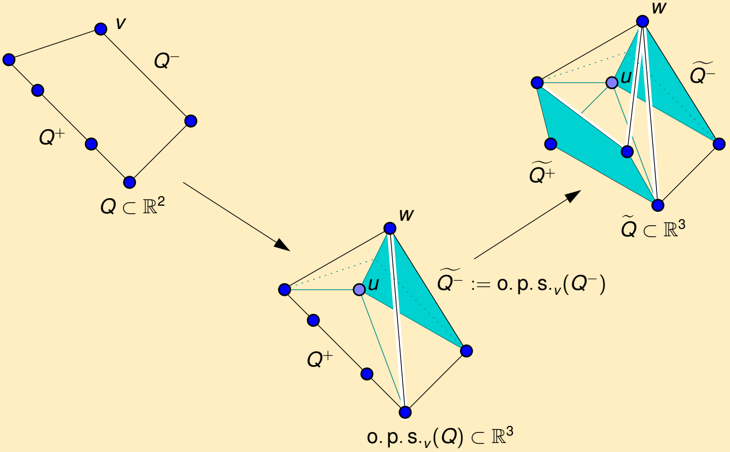

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Theorem (S. 2010)

There is a 5-prismatoid of width 6, with 48 vertices. Hence, there is a non-Hirsch polytope of dimension 43 with 86 facets.

Theorem (Matschke-S.-Weibel 2011)

There is a 5-prismatoid of width 6, with 25 vertices. Hence, there is a non-Hirsch polytope of dimension 20 with 40 facets.

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A 5-pismatoid of width six

Let Q be the polytope having as vertices the 48 rows of the following matrices:

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Theorem

The prismatoid Q of the previous slide has width six.

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There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

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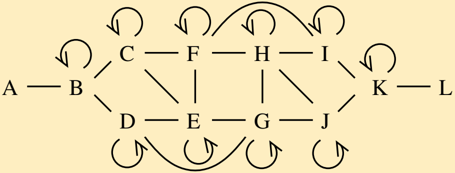
Proof 1 of the Theorem.

It has been verified with `polymake` that the dual graph of Q has the following structure:

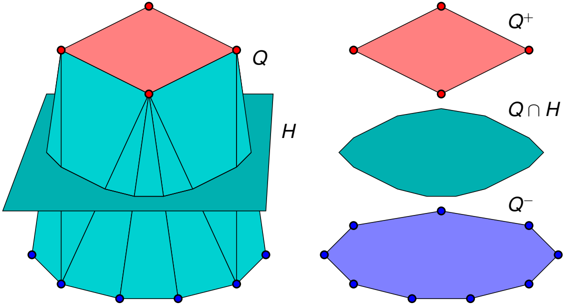


□

Combinatorics of prismatoids

Proof 2 of the Theorem (idea).

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice . . .



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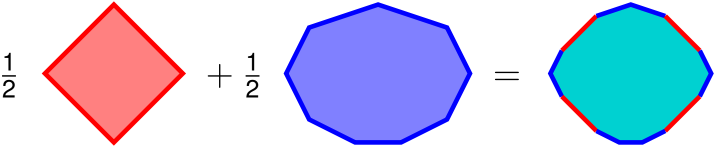
…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.



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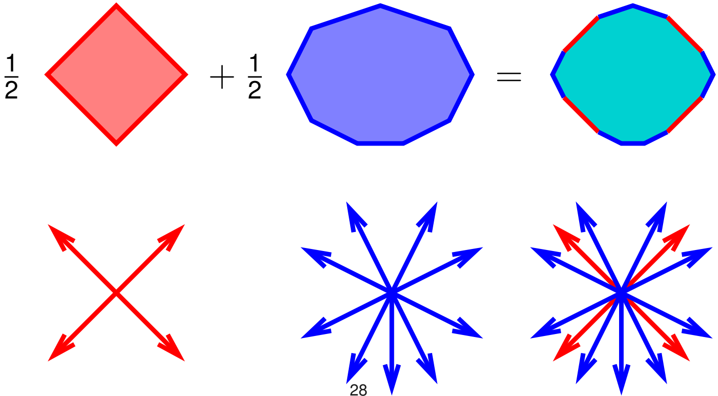
…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-. The normal fan of $Q^+ + Q^-$ equals the "superposition" of those of Q^+ and Q^-.



Combinatorics of prismatoids

So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition (“map”) of the $d-2$-sphere.

Conclusion

4-prismatoids ⇔ pairs of maps in the 2-sphere.
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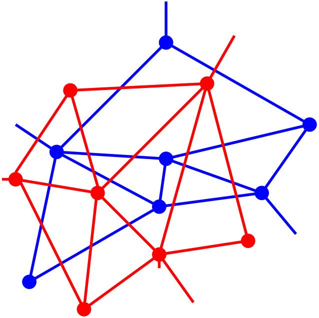

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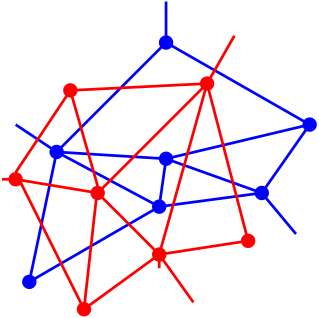

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$\Updownarrow$

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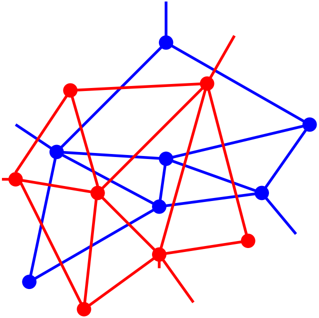

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Replicating the following basic structure we can get a “non-Hirsch” periodic pair of maps in the plane:

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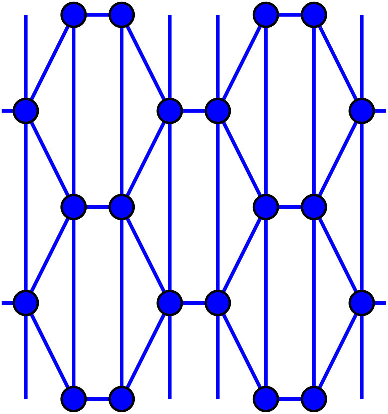

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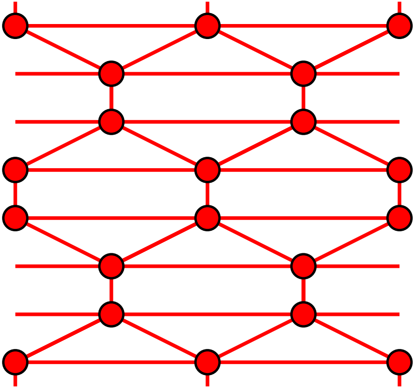

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There are two ways in which a smaller non-Hirsch polytope could be obained:

- Find a smaller 5-prismatoid of width > 5, or
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The latter is impossible:

Theorem (S.-Stephen-Thomas 2011)

Every prismatoid of dimension four has width ≤ 4.

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Corollary

There is a 23-polytope with 46 facets violating the Hirsch conjecture.

Asymptotic width in fixed dimension

If we fix the dimension d, the width of prismatoids is linear:

Theorem

The width of a d-dimensional prismatoid with n vertices cannot exceed $2^{d-3}n$.

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This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. ☐

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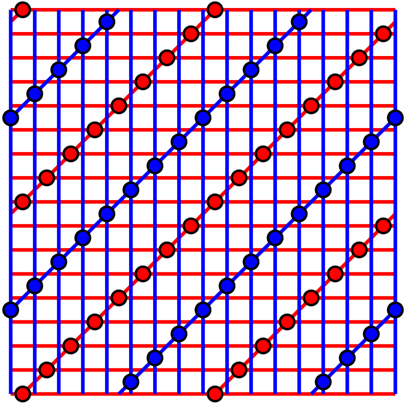

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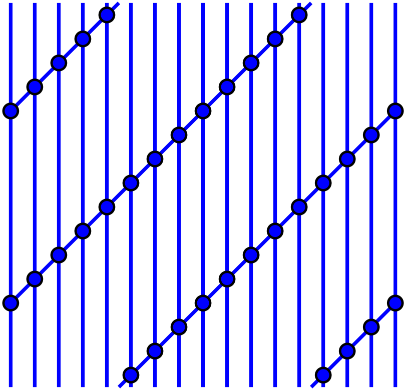

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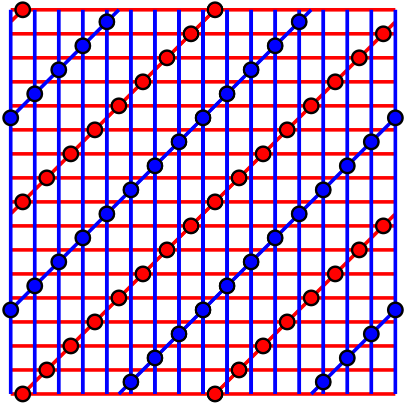

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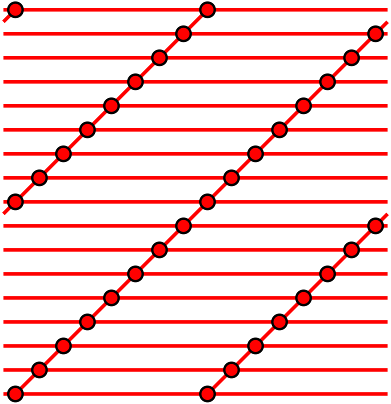

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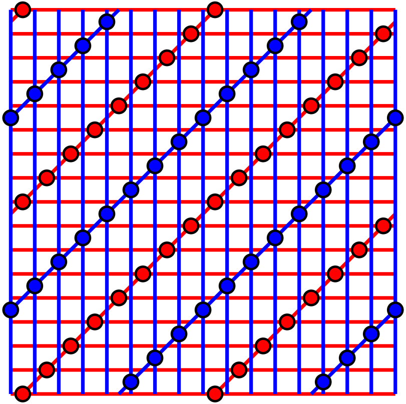

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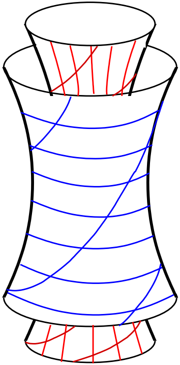

Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps. ☐

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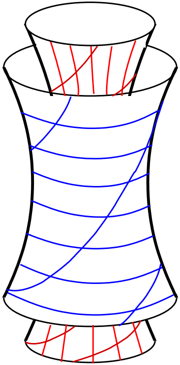

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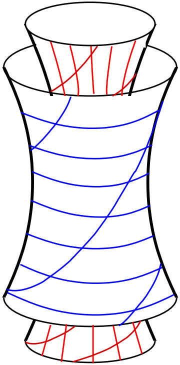

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- Via glueing and products, the counterexample can be converted into an infinite family that violates the Hirsch conjecture by (currently) about 5%.
- This breaks a “psychological barrier”, but for applications it is absolutely irrelevant.

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A proposal for the “next step”:

Conjecture (Hähnle, 2010)

The diameter of a d-polytope with n-facets cannot exceed

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In fact, this conjecture is posed in a much more general setting (*connected layer families*, in the sense of in the sense of Eisenbrand-Hähnle-Razborov-Rothvoss) which would include, for example, all polyhedral manifolds.

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The end

THANK YOU!