LECTURE 7: DIVISORS AND JACOBI INVERSION THEOREM

7.1. **Divisor classes.**

Definition **7.1** *Two divisors D, D' are called linearly equivalent with their difference $D - D' = (f)$ is a principle divisor (=divisor of a meromorphic function). All linearly equivalent divisors belong to the equivalence class, called divisor class, which is labelled as, e.g., $[D]$.*

Since any principal divisor has degree zero, all divisors in the same divisor class have the same degree. Notation for equivalent divisors: $D \equiv D'$.

Definition **7.2** *All Abelian differentials belong to the same divisor class, called the canonical class K.*

This is because the ratio ω_1/ω_2 of any two Abelian differentials is a meromorphic function.

It is an immediate consequence of the Abel theorem that Abel map depends only on the divisor class.

Corollary **7.3.** *All divisors in the same divisor class map to the same point in the Jacobian.*

Proof. From linearity of the Abel map it follows

$$I(D + (f)) = I(D) + I((f)) = I(D),$$

by Abel theorem. □

7.2. **Recap: Riemann-Roch theorem.**

Theorem **7.1.** *For a divisor D on a Riemann surface of genus g*

$$\dim H^0(X, \mathcal{O}(D)) = \deg D - g + 1 + \dim H^1(X, \mathcal{O}(D)).$$

Here

$$H^0(X, \mathcal{O}(D)) = \{f \text{ meromorphic on } X \mid (f) + D \geqslant 0 \text{ or } f \equiv 0\}$$

By Serre duality theorem (t.b.a.) we have

$$H^1(X, \mathcal{O}(D)) \cong H^0(X, \Omega(-D))$$

where

$$H^0(X, \Omega(-D)) := \{\omega \text{ Abelian differential } X \mid (\omega) \geqslant D \text{ or } \omega \equiv 0\}.$$

Notations:

$$h^0(D) = \dim H^0(X, \mathcal{O}(D))$$

($\equiv l(-D)$ can be also encountered),

$$i(D) = \dim H^0(X, \Omega(-D)),$$

the latter is also called index of speciality. Hence

$$h^0(D) = \deg D - g + 1 + i(D).$$

Clearly, dimensions $h^0(D)$ and $i(D)$ depend only on the divisor class. If $D - D' = (f)$ then vector spaces are identified by multiplication by h.

Lemma **7.4.**

$$i(D) = h^0(K - D) \tag{1}$$

Proof. Let ω_0 be an Abelian differential with divisor $(\omega_0) \in [K]$. Then the map $\omega \in H^0(\Omega(-D)) \to \omega/\omega_0 \in H^0(\mathcal{O}(D))$ is an isomorphism of of vector spaces, hence their dimensions are equal. □

This is a consequence of a more profound isomorphism between the corresponding vector spaces, called Hodge duality, which will be covered later on in the course.

7.3. **Canonical class.**

Corollary **7.5.** *The degree of the canonical class* $\deg K = 2g - 2$.

Proof. On the sphere dz has double pole at infinity. $dz = -dw/w^2, z = 1/w$. For $g > 0$ the Rieman-Roch theorem states that $h^0(K) = \deg K - g + 1 + i(K)$. Form Eq. (1) it follows that

$$i(K) = h^0(0) = 1, \quad \text{and} \quad h^0(K) = i(0) = g,$$

the latter is because there are g independent holomorphic differentials. Hence $\deg K = 2g - 2$. □

Proposition **7.6.** *Let X be a compact Riemann surface. If there is a meromorphic function on X having exactly one pole, and that pole has order one, then X is biholomorphic to the Riemann sphere.*

Proof. Let $F : X \to P^1$ be the given meromorphic function. The hypotheses imply that the degree (i.e., number of points in the preimage) of F is 1 (computing using the preimage of $\infty \in P^1$ consists of the one pole of order one). This means that for any $y \in P^1$ there is exactly one point x in the preimage of y (and the multiplicity of F is one there). Thus F is a bijection. The inverse map is holomorphic since the derivative of F is bijective, so F has a local holomorphic inverse, which coincides with the global inverse F^{-1}. □

More on branched coverings in upcoming lectures.

Corollary **7.7.** *There is no point on X where all holomorphic differentials vanish simultaneously.*

Proof. Suppose there exists such a point $P \in X$. Then for divisor $D = P$, we have $i(P) = g$ and from the Riemann-Roch theorem it follows $h^0(D) = 2$. Then besides the constant function, there exists a nontrivial meromorphic function with only one simple pole at P. Due to the previous Proposition X is biholomorphic to the sphere. □

7.4. **The Abel map as an embedding.**

Definition **7.8** *A holomorphic map $F : X \to Y$ between complex manifolds is called embedding if F is an immersion (derivative is injective at every point) and $F : X \to F(X)$ is an homeomorphism.*

Lemma **7.9.** *If X is compact, this is equivalent to F being injective immersion.*

Proof. Indeed, then $F : X \to F(X)$ is bijective and continuous. We will use the following fact: a function g is continuous iff $g^{-1}(C)$ is closed for all C closed in X. Take $g = F^{-1}$, then $g^{-1}(C) = F^{-1^{-1}}(C) = F(C)$. C closed in X compact means C is compact. Since F is continuous, it follows that $F(C)$ is compact. Hence $F(C)$ is closed. Hence F^{-1} is continuous. Hence $F : X \to F(X)$ is a homeomorphism. □

Lemma **7.10.** *The Abel map $I(P) = \int_{P_0}^{P} \omega_j$ is an embedding.*

Proof. Derivative of the Abel map at a point P equals

$$dI(P) = \omega_j(P)$$

From Cor. 7.7 we know that for any point $P \in X$ holomorphic differentials cannot vanish at P, hence $dI(P) \neq 0$, so the Abel map is an immersion.

Suppose that two point $P_1, P_1 \in X$ have the same image $I(P_1) = I(P_2)$. Then $I(P_1 - P_2) \equiv 0$ and by Abel theorem $P_1 - P_2$ is a principle divisor. By Prop. 7.6 meromorphic function with just one simple pole does not exist for $g > 0$, hence $P_1 = P_2$. □

7.5. **Jacobi inversion theorem.** The set X_n of positive divisors of degree n can be described as nth symmetric product of X with itself, $X_n = X \times ... \times X/Sym_n$, where quotient by the symmetric group Sym_n means that we do not distinguish between the points.

In what follows we will also need a notion of a special divisor.

Definition **7.11** *A positive divisor D of degree g is called special if $i(D) > 0$.*

(Hence the name index of speciality). In other words there exists a holomorphic differential ω with divisor

$$(\omega) \geqslant D. \tag{2}$$

This is rare. Indeed, since holomorphic differentials form a dim-g vector space, we can write $\omega = \sum \alpha_j \omega_j$ for some basis. Then Eq. (2) translates into a homogeneous system of linear equations on coefficients α_j, one equation for each zero. So most positive divisors are non-special. In particular, in the proof of next theorem we will see that in for every non-special divisor there is a neighbourhood where all divisors are also non-special.

THEOREM **7.2. (Jacobi inversion theorem)** *Consider the set X_g of positive divisors of degree g. The Abel map*

$$I: \; X_g \to Jac(X)$$

on this set is surjective.

Proof. We should show that for any point $C_j \in Jac(X)$ there exists a positive divisor $D = P_1 + ... + P_g$ of degree g, such that

$$C_j = \sum_{l=1}^{g} \int_{P_0}^{P_l} \omega_j.$$

Let us start with some non-special divisor $D_z = z_1 + ... + z_g$. Consider the Abel map for this divisor $I(D_z)$ and compute its differential

$$\frac{d}{dz_l} I(D_z) = \frac{\omega_j(z_l)}{dz_l}.$$

Hence the Jacobian matrix of the map is

$$\begin{pmatrix} \frac{\omega_1(z_1)}{dz_1} & \dots & \frac{\omega_1(z_g)}{dz_g} \\ \vdots & & \vdots \\ \frac{\omega_g(z_1)}{dz_1} & \dots & \frac{\omega_g(z_g)}{dz_g} \end{pmatrix} \tag{3}$$

□

By the assumption that D_z is a non-special divisor, the determinant of this matrix is non-zero. Therefore, by the implicit function theorem I maps the neighbourhood of $(z_1, ..., z_g)$ bijectively onto a neighbourhood $V_{I(D_z)} \subset Jac(X)$ of the point $I(D_z) \in Jac(X)$.

Now let $C_j \in Jac(X)$ be an arbitrary point. One can always find $n \in \mathbb{N}$ big enough so that

$$I(D_z) + \frac{1}{n} C \in V_{I(D_z)}$$

Then there exists another non-special divisor D_w (in the vicinity of D_z) such that it is the preimage of the point above.

$$I(D_w) = I(D_z) + \frac{1}{n} C.$$

Then

$$C = n(I(D_w) - I(D_z))$$

and we need to show that

$$C = I(D), \quad \text{where } D = P_1 + ... + P_g \text{ is a positive divisor of } \deg g.$$

Consider the divisor

$$D' = n \sum_{j=1}^{g} w_j - n \sum_{j=1}^{g} z_j + gP_0$$

of degree g. By Riemann-Roch theorem,

$$h^0(D') = g + 1 - g + i(D') \geqslant 1.$$

Hence there exists a meromorphic function f with divisor $(f) + D' \geqslant 0$. Hence $(f) + D'$ is a positive divisor of degree g and we can write for this divisor

$$D = P_1 + ... + P_g = (f) + D'$$

Applying Abel theorem for this divisor we get

$$I(D) = I(n_w D - nD_z + gP_0 - gP_0) = C.$$