Complex Geometry - Homework 1

1. Problem

Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ be two open sets, $F = (F_1, \ldots, F_m)\colon D_1 \to D_2$ a C^1 map and $g\colon D_2 \to \mathbb{C}$ a C^1 function. Prove the following:

a) Given a C^1 curve $\gamma = (\gamma_1, \ldots, \gamma_m)\colon I \to D_2$ defined on some open interval $I \subset \mathbb{R}$ one has that

$$\frac{\partial (g \circ \gamma)}{\partial t}(t) = \sum_{j=1}^{m} \left[\frac{\partial g}{\partial w_j}(\gamma(t)) \cdot \frac{\partial \gamma_j}{\partial t}(t) + \frac{\partial g}{\partial \overline{w}_j}(\gamma(t)) \cdot \overline{\frac{\partial \gamma_j}{\partial t}(t)} \right]$$

holds for all $t \in I$.

b) One has

$$\frac{\partial (g \circ F)}{\partial z_j} = \sum_{j=1}^{m} \left[\left(\frac{\partial g}{\partial w_k} \circ F \right) \frac{\partial F_k}{\partial z_j} + \left(\frac{\partial g}{\partial \overline{w}_k} \circ F \right) \overline{\left(\frac{\partial F_k}{\partial \overline{z}_j} \right)} \right].$$

2. Problem

Let $f\colon \mathbb{C}^n \to \mathbb{C}^m, f = (f_1, \ldots, f_m)$ be a map of class C^1. Write $f_j = u_j + \sqrt{-1} v_j$ and consider $f = (u_1, v_1, \ldots u_m, v_m)$ as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. Denote by

$$J_f^{\mathbb{R}}(z) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial y_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial v_m}{\partial x_1} & \frac{\partial v_m}{\partial y_1} & \cdots & \frac{\partial v_m}{\partial y_n} \end{pmatrix}$$

the real Jacobi matrix of f. Furthermore, set

$$J_f(z) = \left(\frac{\partial f_i}{\partial z_j}(z) \right)_{\substack{1 \le i \le m, \\ 1 \le j \le n}}.$$

a) We have an $\mathbb{R}$-linear isomorphism $T_n\colon \mathbb{R}^{2n} \to \mathbb{C}^n,\ (x_1, y_1, \ldots, x_n, y_n) \mapsto (x_1 + \sqrt{-1} y_1, \ldots, x_n + \sqrt{-1} y_n)$ between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ seen as real vector space. Then $T_m J_f^{\mathbb{R}}(z) T_n^{-1}$ defines an $\mathbb{R}$-linear map between $\mathbb{C}^n$ and $\mathbb{C}^m$. Show that f is holomorphic on D if and only if $T_m J_f^{\mathbb{R}}(z) T_n^{-1}$ is $\mathbb{C}$-linear for all $z \in D$.

b) Assuming that f is holomorphic, show that the transformation matrix of $T_m J_f^{\mathbb{R}}(z) T_n^{-1}$ in the canonical bases of $\mathbb{C}^n$ and $\mathbb{C}^m$ seen as complex vector spaces is given by $J_f(z)$.

c) Asumme $n = m$ and that f is holomorphic. Show that

$$\det J_f^{\mathbb{R}}(z) = |\det J_f(z)|^2 .$$

d) Asumme $n = m$. Show that

$$\det J_f^{\mathbb{R}}(z) = \det \begin{pmatrix} J_f(z) & \overline{J_{\bar{f}}(z)} \\ J_{\bar{f}}(z) & \overline{J_f(z)} \end{pmatrix} .$$

e) Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ be open sets and $f\colon D_1 \to D_2$, $g\colon D_2 \to \mathbb{C}^\ell$ be holomorphic maps. Show that $g \circ f$ is holomorphic with

$$J_{g\circ f}(z) = J_g(f(z)) J_f(z), \qquad z \in D_1.$$

3. Problem

Let $D \subset \mathbb{C}^n$ be a polydisc of multiradius $r = (r_1, \dots, r_n)$ centred in $a \in \mathbb{C}^n$, $f \in \mathcal{O}(D)$ a holomorphic function and $\alpha \in \mathbb{N}^n$. Verify **Cauchy's Estimates**:

$$(i) \qquad |\partial^\alpha f(a)| \le \frac{\alpha!}{r^\alpha} \sup_{z\in D} |f(z)|,$$

$$(ii) \qquad |\partial^\alpha f(a)| \le \frac{\alpha!(\alpha_1+2)\cdots(\alpha_n+2)}{(2\pi)^n r_1^{\alpha_1+2}\cdots r_n^{\alpha_n+2}} \int_D |f(z)| dV_{\mathbb{C}^n},$$

where $\alpha \in \mathbb{N}^n$ and $\partial^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$.

4. Problem

A function $f \in \mathcal{O}(\mathbb{C}^n)$ is called an entire (holomorphic) function. Prove **Liouville's Theorem**: Every bounded entire function is constant.

Let f be an entire function, and suppose that there exist a multi-index α and a constant $C > 0$ such that $|f(z)| \leqslant C|z^\alpha|$ for every $z \in \mathbb{C}^n$. Show that f is a polynomial of degree at most $|\alpha|$.

5. Problem

Let $D \subset \mathbb{C}^n$ be an open set and $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{O}(D)$ a sequence of holomorphic functions which converges locally uniformly to some function $f\colon D \to \mathbb{C}$. Show that f is holomorphic on D and $\partial^\alpha f_k \to \partial^\alpha f$ locally uniformly on D for $k \to \infty$.

6. Problem

Let $f(z) = \sum_{\nu\in\mathbb{N}^n} c_\nu z^\nu$ be a power series with non empty domain of convergence D. Show that

a) f defines a holomorphic function $f \in \mathcal{O}(D)$,

b) $\sum_{\nu \in \mathbb{N}^n} c_\nu \left(\partial^\alpha z^\nu\right)$ converges to $\partial^\alpha f$ locally uniformly on D,

c) $c_\alpha = \frac{1}{\alpha!}\partial^\alpha f(0)$ for any $\alpha \in \mathbb{N}^n$.

7. Problem
Given $r = (r_1, \ldots, r_n) \in (0,1)^n$, $n \geq 2$, consider the set $H(r) \subset \mathbb{C}^n$ defined by

$$\begin{aligned} H(r) = &\{z \in \mathbb{C}^n \mid |z_j| < 1 \text{ for } j < n, \quad r_n < |z_n| < 1\} \\ &\cup \{z \in \mathbb{C}^n \mid |z_j| < r_j \text{ for } j < n, \quad |z_n| < 1\} \end{aligned}$$

and let $D \subset \mathbb{C}^n$ be the polydisc of radius 1 centred in 0.

a) Show that $H(r) \subset D$ is a domain, that is $H(r)$ is open and connected.

b) Prove that any holomorphic function $f \in \mathcal{O}(H(r))$ has a unique extension $\tilde{f} \in \mathcal{O}(D)$.

c) Let U be an open neighbourhood of ∂D such that $U \cap D$ is connected (∂D denotes the boundary of $D \subset \mathbb{C}^n$). Show that $f \in \mathcal{O}(U)$ extends holomorphically to $D \cup U$.

d) Let $U \subset \mathbb{C}^n$ be open, $a \in U$ a point and $f \in \mathcal{O}(U \setminus \{a\})$. Show that f has a holomorphic extension on U.

e) Are c) and d) valid when $n = 1$?