WS 16/17 Marinescu / Herrmann

Complex Geometry - Homework 2

1. Problem

a) Show that $d\overline{w} \wedge dw = 2i dv_{\mathbb{C}}$, where $dv_{\mathbb{C}} = dx \wedge dy$ denotes the standard volume form on $\mathbb{C} \simeq \mathbb{R}^2$.

b) Let $U \subset \mathbb{C}$ be a bounded domain and $f \in C^0(\overline{U})$. Show that the integral

$$\int_U (z-w)^{-1} f(w) d\overline{w} \wedge dw$$

exists for any $z \in \mathbb{C}$ and if $B_\varepsilon(z) \subset U$ we have

$$\frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z} dw = f(z) + o(\varepsilon), \ \ \varepsilon \to 0,$$

$$\frac{1}{2\pi i} \int_{|w-z|\leq\varepsilon} \frac{f(w)}{w-z} d\overline{w} \wedge dw = O(\varepsilon), \ \ \varepsilon \to 0.$$

c) Assume that U has piecewise C^1-boundary and let $a \in C^1(\overline{U})$. Use Stokes formula and the decomposition $d = \partial + \overline{\partial}$ to prove that

$$\int_{\partial U} a \, dw = \int_U \frac{\partial a}{\partial \overline{w}} \, d\overline{w} \wedge dw.$$

d) Let $u \in C^1(\overline{U})$ and $z \in U$. For $B_\varepsilon(z) \subset U$ apply the previous formula on $U \setminus B_\varepsilon(z)$ for $a(w) = u(w)(w-z)^{-1}$ and let $\varepsilon \to 0$ to deduce the **Cauchy-Pompeiu formula**:

$$u(z) = \frac{1}{2\pi i}\left(\int_{\partial U} \frac{u(w)}{w-z} dw + \int_U \frac{1}{w-z}\frac{\partial u}{\partial \overline{w}}(w) dw \wedge d\overline{w}\right). \tag{1}$$

d) Calculate the value of the left-hand expression in (1) for $z \in \mathbb{C} \setminus \overline{U}$.

2. Problem

Let $f \in C_0^k(\mathbb{C})$, $k \geq 1$, be a k-times continuously differentiable function with compact support. Show that there exists $u \in C^k(\mathbb{C})$ such that $\overline{\partial} u = f d\overline{z}$ on $\mathbb{C}$. (Hint: apply (1) for an open set U with $\operatorname{supp} f \subset U$.) Show that if f depends smoothly or holomorphically on parameters we can choose the solution u to be also smooth or holomorphic in these parameters.

3. Problem
Let $g \in \Omega_0^{0,1}(\mathbb{C})$ a smooth $(0,1)$-form with compact support. Show that there exists $f \in C_0^\infty(\mathbb{C})$ with $\overline{\partial} f = g$ if and only if $\int_{\mathbb{C}} g \wedge h dz = 0$ for any $h \in \mathcal{O}(\mathbb{C})$.

4. Problem
Let $D \subset \mathbb{C}^n$ be an open set and $U \Subset D$ be a relatively compact polydisc in D. Let $p, q \geq 0$. Our purpose is to show the following assertion, known as the **Dolbeault-Grothendieck Lemma**:

$$\forall f \in \Omega^{p,q+1}(D) \ \overline{\partial} f = 0 \ \exists\, u \in \Omega^{p,q}(U) : \ \overline{\partial} u = f \tag{2}$$

a) Show that we can reduce the problem to the case $p = 0$. Use that any $f \in \Omega^{p,q}(D)$ can be written uniquely as $f = \sum_{|I|=q, I\uparrow} f_I dz_I$ with $f_I \in \Omega^{0,q}(D)$.

b) If $k \in \mathbb{N}$ denote by $\Omega_k^{0,\bullet}(D)$ the sub-algebra of $\Omega^{0,\bullet}(D)$ generated by $d\overline{z}_1, \dots, d\overline{z}_{k-1}$ and $C^\infty(D)$. Determine $\Omega_1^{0,q}(D)$ and $\Omega_{n+1}^{0,q}(D)$.

c) Show that any $f \in \Omega_{k+1}^{0,q+1}(D)$ can be written as $f = d\overline{z}_k \wedge g + h$ with $g \in \Omega_k^{0,q}(D)$ and $h \in \Omega_k^{0,q+1}(D)$. Show that if $\overline{\partial} f = 0$ the coefficients of g and h are holomorphic in the variables $z_{k+1}, \dots, z_n$.

d) We prove (2) by induction with respect to k. Show that (2) is true for $f \in \Omega_1^{p,q+1}(D)$ and assume that it is true for all $f \in \Omega_k^{p,q+1}(D)$ for some $k \in \mathbb{N}$. Let $f \in \Omega_{k+1}^{0,q+1}(D)$, $\overline{\partial} f = 0$ and write $f = d\overline{z}_k \wedge g + h$ with $g \in \Omega_k^{0,q}(D)$ and $h \in \Omega_k^{0,q+1}(D)$. Use the solution of the $\overline{\partial}$-equation in Problem 2 to find $\tilde{g} \in \Omega_{k+1}^{0,q}(D)$ with $\partial \tilde{g} / \partial \overline{z}_k = g$ on U and whose coefficients are holomorphic in $z_{k+1}, \dots, z_n$. (We denote by $\partial \tilde{g} / \partial \overline{z}_k$ the form obtained by taking the derivative $\partial / \partial \overline{z}_k$ of each coefficient of $\tilde{g}$.) Show that $f - \overline{\partial} \tilde{g} \in \Omega_k^{0,q+1}(D)$ and conclude.

5. Problem
Let $D \subset \mathbb{C}$ be an open disc.

a) Given $f \in C^\infty(D)$, show that there is $u \in C^\infty(D)$ with $\partial u / \partial \overline{z} = f$. For this purpose choose a sequence $(D_\nu)_{\nu \in \mathbb{N}}$ of open discs with $\overline{D_\nu} \subset D_{\nu+1}$ for all $\nu \in \mathbb{N}$ and $\bigcup_{\nu \in \mathbb{N}} D_\nu = D$. Show that there exists $u_\nu \in C^\infty(D)$ with $\partial u_\nu / \partial \overline{z} = f$ on D_ν and we can assume $|u_\nu - u_{\nu-1}| \leq 2^{-\nu}$ on $D_{\nu-1}$. Deduce that the sequence $(u_\nu)_{\nu \in \mathbb{N}}$ converges to a limit u which solves the equation.

b) Describe the Dolbeault cohomology groups of D.

6. Problem
onsider $P = \Pi_{j=1}^n D_j$, where $D_j \subset \mathbb{C}$ is an open disc or $D_j = \mathbb{C}$.

a) Describe $H^{p,0}(P)$ for $p \geq 0$.

b) Show that $H^{p,q}(P) = 0$ for $q \geq 2$. Let $\alpha \in \Omega^{p,q}(P)$ with $\overline{\partial}\alpha = 0$. Let $(P_\nu)_{\nu\in\mathbb{N}}$ be a sequence of polydiscs of $\mathbb{C}^n$ with $\overline{P_\nu} \subset P_{\nu+1}$ for all $\nu \in \mathbb{N}$ and $\bigcup_{\nu\in\mathbb{N}} P_\nu = P$. Deduce from the Dolbeault-Grothendieck lemma that there exists a sequence $(\beta_\nu)_{\nu\in\mathbb{N}}$ of $\Omega^{p,q-1}(P)$ such that $\overline{\partial}\beta_\nu = \alpha$ on P_ν and $\beta_{\nu+1} = \beta_\nu$ on P_ν. Conclude.

c) Show that $H^{0,1}(P) = 0$ by proceeding as in the problem on the Dolbeault cohomology of the disc in $\mathbb{C}$.

7. Problem
Set $U = \mathbb{C}^2 \setminus \{0\}$. We show $H^{0,1}(U) \neq 0$.

a) Using the relation
$$\frac{1}{z_1 z_2} = \frac{\overline{z_2}}{z_1 r^2} + \frac{\overline{z_1}}{z_2 r^2}$$
on $\{z_1 z_2 \neq 0\}$ where $r = \|z\|_2$, deduce that there exists $\omega \in \Omega^{0,1}(U)$ with $\overline{\partial}\omega = 0$ and $\omega = \overline{\partial}(\overline{z_2}/(z_1 r^2))$ on $\{z_1 \neq 0\}$.

b) Assume there exists $f \in C^\infty(U)$ with $\overline{\partial} f = \omega$ on U. Show that the function $u(z) = z_1 f(z) - \overline{z}_2 r^{-2}$ extends to a holomorphic function on $\mathbb{C}^2$. Conclude.