Complex Geometry - Homework 3

1. Problem
Prove the holomorphic implicit mapping theorem by using the implicit mapping theorem from real analysis.

2. Problem
Let $f\colon X \to Y$ be a holomorphic submersion between complex manifolds. Show that if M is a complex submanifold of Y, then $f^{-1}(M)$ is a complex submanifold of X. What is $\dim_{\mathbb{C}} f^{-1}(M)$?

3. Problem
Let $q \geq 1$ be an integer and $\nu(0), \ldots, \nu(N)$, $N+1 = \binom{n+q}{q}$ be an ordering of the set $\{\nu \in \mathbb{N}_0^{n+1} \mid |\nu| = q\}$. Consider the map $\tilde{\varphi}\colon \mathbb{C}^{n+1} \to \mathbb{C}^{N+1}$, $z = (z_0, \ldots, z_n) \mapsto (z^{\nu(0)}, \ldots, z^{\nu(N)})$. Prove the following.

a) The map $\tilde{\varphi}$ defines a holomorphic map
$$\varphi\colon \mathbb{P}^n \to \mathbb{P}^N, \quad [z_0, \ldots, z_n] \mapsto [z^{\nu(0)}, \ldots, z^{\nu(N)}],$$
which is a homeomorphism on its image (called *Veronese embedding*).

b) The map φ is an immersion and $\varphi(\mathbb{P}^n)$ is a compact submanifold of $\mathbb{P}^N$.

c) $\varphi(\mathbb{P}^n) = \{[w] \in \mathbb{P}^N \mid w_\nu w_\mu - w_{\nu'} w_{\mu'} = 0, \nu + \mu = \nu' + \mu'\}$.

Let $H = \{[z] \in \mathbb{P}^n \mid f(z) = 0\}$ a hypersurface in $\mathbb{P}^n$ where $f \in \mathbb{C}[z_0, \ldots, z_n]$ is a homogeneous polynomial of total degree q. Show that:

d) The set $\varphi(H)$ is the intersection of $\varphi(\mathbb{P}^n)$ with a hyperplane in $\mathbb{P}^N$.

e) If $X \subset \mathbb{P}^n$ is a projective algebraic variety, then $X \setminus H$ is biholomorphic to an affine algebraic variety.

4. Problem
Consider the map
$$\psi\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}, \quad ([z_0, \ldots, z_n], [w_0, \ldots, w_m]) \mapsto [x_{00}, x_{01}, \ldots, x_{nm}]$$
with $x_{jk} = z_j w_k$. Prove the following.

a) The map ψ is an embedding (called *Segre embedding*).

b) $\psi(\mathbb{P}^n \times \mathbb{P}^m) = \{[x] \in \mathbb{P}^{nm+n+m} \mid x_{jp}x_{kq} - x_{jq}x_{kp} = 0, 0 \leq j,k \leq n, 0 \leq p,q \leq m\}$.

c) $\pi^{-1}(\psi(\mathbb{P}^n \times \mathbb{P}^m))$ is a submanifold, where $\pi\colon \mathbb{C}^{nm+n+m+1} \setminus \{0\} \to \mathbb{P}^{mn+n+m}$, $x \mapsto [x]$, denote the canonical projection.

d) Every finite product of projective algebraic varieties is projective-algebraic.

5. Problem
Consider the equivalence relation on $\mathbb{C}^{2^*} := \mathbb{C}^2 \setminus \{0\}$, $z \sim w \Leftrightarrow \exists j \in \mathbb{Z} : 2^j z = w$. The quotient $M = \mathbb{C}^{2^*}/\sim$ is called Hopf surface.

a) Verify that the Hopf surface is a complex manifold.

b) Show that the map $\varphi\colon \mathbb{R} \times S^3 \to \mathbb{C}^{2*}$, $(t, z) \mapsto 2^t z$, is a homeomorphism which satisfies $\varphi(t+j, z) = 2^j \varphi(t, z))$ where $S^3 = \{z \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$.

c) Use $S^1 = \mathbb{R}/\mathbb{Z}$ to prove that M is homeomorphic to $S^1 \times S^3$ and conclude that it is compact.

Remark: It can be shown that the Hopf surface is not projective-algebraic.

6. Problem

Fix $\zeta = e^{i2\pi/5}$.

a) Show that $([m], [z_j]) \mapsto [\zeta^{jm} z_j]$ defines a holomorphic group action of $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^3$ and describe the set of fixed points.

b) Verify that $M = \{[z] \in \mathbb{P}^3 \mid z_0^5 + \ldots + z_3^5 = 0\}$ is preserved under that group action.

c) Prove that the quotient $M/\mathbb{Z}_5$ (called Godeaux surface) is a compact complex manifold.

7. Problem

The space $G_k(n) = M_k(n)/GL(k, \mathbb{C})$ is a compact manifold (called Grassman manifold) where

$$M_k(n) = \{A \in \mathrm{Mat}_{k\times n}(\mathbb{C}) \simeq \mathbb{C}^{k\times n} \mid \mathrm{rank}(A) = k\}$$

and $GL(k, \mathbb{C})$ acts on $M_k(n)$ via left multiplication. It can be seen as the set of all k-dimensional subspaces of $\mathbb{C}^n$.

a) Show that $\{(U_\nu, \varphi_\nu = \psi_\nu^{-1}) \mid \nu \in \mathbb{N}^k, 1 \le \nu_1 < \ldots < \nu_k \le n\}$ is a holomorphic atlas on $G_k(n)$ where $U_\nu = \psi_\nu(\mathbb{C}^{k(n-k)})$ and ψ_ν is defined by

$$\psi_\nu\colon \mathbb{C}^{k(n-k)} \to G_k(n), \quad \psi(v_1, \ldots, v_{n-k}) = [w_1, \ldots, w_n]$$

with

$$w_j = \begin{cases} e_l & , \text{ if } j = \nu_l, \\ v_l & , \text{ if } j = \mu_l, \end{cases}$$

where $\mu \in \mathbb{N}^{n-k}$, $1 \le \mu_1 < \ldots < \mu_{n-k} \le n$, $\mu_j \in \mathbb{N} \setminus \{\nu_1, \ldots, \nu_k\}$ is the complementary index of ν and $e_l = (0, \ldots, 0, 1, 0 \ldots, 0)^T \in \mathbb{C}^k$ has its only non-zero entry at position l.

b) Set $N = \binom{n}{k}$. Show that the map $\tilde{\Phi}\colon M_k(n) \to \bigwedge^k \mathbb{C}^n \simeq \mathbb{C}^N$, $A = (A_1, \ldots, A_k)^T \mapsto A_1 \wedge \ldots \wedge A_k$ defines a holomorphic map $\Phi\colon G_k(n) \to \mathbb{P}^{N-1}$.

c) Show that Φ is an embedding (called *Plücker embedding*).

d) Conclude that $G_k(n)$ is projective-algebraic.

8. Problem

Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Gamma' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ be two lattices in $\mathbb{C}$, i.e. $\omega_j, \omega_j' \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ for $j \in 1, 2$ and $\mathrm{Im}\frac{\omega_1}{\omega_2} \neq 0$ resp. $\mathrm{Im}\frac{\omega_1'}{\omega_2'} \neq 0$. Show that $\Gamma = \Gamma'$ if and only if

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$$

for some matrix $A \in \mathrm{SL}(2, \mathbb{C}) := \{A \in \mathrm{GL}(2, \mathbb{C}) \mid \det A = 1\}$.

9. Problem

(a) Let $\Gamma, \Gamma' \subset \mathbb{C}$ be two lattice and $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$. Show that the map $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \alpha z$, induces a holomorphic map $\tilde{f} : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ which satisfies

$$\tilde{f} \circ \pi = \pi' \circ f$$

where $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ resp. $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ are the standard projection maps. Furthermore, show that $\tilde{f}$ is biholomorphic if and only if $\alpha\Gamma = \Gamma'$.

(b) Show that every torus $X = \mathbb{C}/\Gamma$ (seen as additive group) is isomorphic to a torus of the form

$$X(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$.

(c) Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ and $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$. Let

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Show that the tori $X(\tau)$ and $X(\tau')$ are isomorphic.