Complex Geometry - Homework 4

1. Problem

For $n \in \mathbb{N}$, $k \in \mathbb{Z}$, consider the holomorphic line bundle $\mathcal{O}(k) \to \mathbb{P}^n$ defined by $\mathcal{O}(k) = J^k$ for $k < 0$, $\mathcal{O}(0) = \mathbb{C}$, $\mathcal{O}(k) = (J^*)^k$ for $k > 0$, where $J \to \mathbb{P}^n$ is the tautological line bundle. Set $I = \{0, \dots, n\}$. For $j, \ell \in I$ let $U_j \subset \mathbb{P}^n$ defined by

$$U_j = \{[z_0, \dots, z_n] \in \mathbb{P}^n \mid z_j \neq 0\}$$

and set $g_{\ell,j} \colon U_j \cap U_\ell \to \mathbb{C}$, $g_{\ell,j}([z_0, \dots, z_n]) = \left(\frac{z_j}{z_\ell}\right)^k$.

a) Show that $(\{U_j\}_{j\in I}, \{g_{\ell,j}\}_{\ell,j\in I})$ is a cocycle associated to $\mathcal{O}(k)$.

b) Show that the space of global holomorphic sections with values in $\mathcal{O}(k)$ can be identified with the space of homogeneous polynomials in $n+1$ variables of total degree equal to k, that is

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \simeq \left\{ \sum_{\alpha \in \mathbb{N}_0^{n+1},\, |\alpha| = k} a_\alpha z^\alpha \right\} \subset \mathbb{C}[z_0, \dots, z_n].$$

and to the space of complex polynomials in n variables of total degree less or equal to k, that is

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \simeq \left\{ \sum_{\alpha \in \mathbb{N}_0^{n},\, |\alpha| \leq k} a_\alpha z^\alpha \right\} \subset \mathbb{C}[z_1, \dots, z_n].$$

c) Calculate the dimension of $H^0(\mathbb{P}^n, \mathcal{O}(k))$.

2. Problem

Let $K_{\mathbb{P}^n}$ be the canonical bundle of $\mathbb{P}^n$. Prove that $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.

3. Problem

Consider $U_0 = \mathbb{C} \setminus \{0\}$, $U_1 = \mathbb{C} \setminus \{1\}$, $g_{1\,0}, g_{0\,1} \colon U_0 \cap U_1 \to \mathbb{C}$, $g_{1\,0}(z) = z(1-z)$ and $g_{0\,1}(z) = (g_{1\,0}(z))^{-1}$.

a) Show that $(\{U_j\}, \{g_{\ell,j}\})$ defines a holomorphic line bundle over $\mathbb{C}$ (denoted by L) and describe the space of holomorphic sections $H^0(\mathbb{C}, L)$.

b) Show that L is isomorphic to the trivial line bundle $\pi \colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$, $\pi(z, \lambda) = z$. (*Hint: construct a non vanishing section $s \in H^0(\mathbb{C}, L)$.*)

4. Problem

Let V be a real vector space of finite dimension and $J \in \operatorname{End}(V)$ be an endomorphism satisfying $J^2 = -\operatorname{Id}$. Verify the following properties

i) J is invertible

ii) There exists an inner product on V with $\langle Jv, Jw\rangle = \langle v, w\rangle$ for all $v, w \in V$

iii) There exists an Hermitian inner product on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ such that $i \otimes J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is selfadjoint.

iv) $V_{\mathbb{C}} = \operatorname{Eig}(1 \otimes J, i) \oplus \operatorname{Eig}(1 \otimes J, -i)$ with $\overline{\operatorname{Eig}(1 \otimes J, i)} = \operatorname{Eig}(1 \otimes J, -i)$.

v) $\dim_{\mathbb{R}} V = 2n$ for some $n \in \mathbb{N}$ and it exists $T \in GL(V)$ with

$$TJT^{-1} = \begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}.$$

vi) $P = \frac{1}{2}(1 \otimes Id - i \otimes J) : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is a projection (i.e $P \circ P = P$) with $\operatorname{ran} P = \operatorname{Eig}(1 \otimes J, i)$ and $\ker P = \operatorname{Eig}(1 \otimes J, -i)$.

5. Problem

Let X be a smooth real manifold of dimension $\dim_{\mathbb{R}} X = 2n$.

a) Consider $2n \times 2n$-matrix

$$J_n = \begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}.$$

Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $x_j + ix_{n+j} = z_j$. Given a map $f\colon \mathbb{C}^n \to \mathbb{C}^m$ denote by $\tilde{f}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ the corresponding real map. Show that f is holomorphic if and only if $d\tilde{f} J_n = J_m d\tilde{f}$ holds.

b) An almost complex structure on X is a section $J \in \Gamma(\operatorname{End}(TX))$ with $J_x^2 = -Id_{T_xX}$ for all $x \in X$. Given such J we set $V_x^J = \operatorname{Eig}(1 \otimes J_x, i)$. Show that $V^J = \bigsqcup_{x\in X} V_x^J$ is a complex subbundle of $\mathbb{C} \otimes TX$ of rank n, which satisfies $\mathbb{C} \otimes TX = V^J \oplus \overline{V^J}$.

c) Assume that $V \subset \mathbb{C} \otimes TX$ is a complex subbundle of rank n with $\mathbb{C} \otimes TX = V \oplus \overline{V}$. Show that there exists an almost complex structure J on X satisfying $V^J = V$.

d) Given a holomorphic atlas on X, show that there exists an almost complex structure J on X such that

   i) $f : X \to \mathbb{C}$ is holomorphic if and only if $d\tilde{f} J_n = J_1 d\tilde{f}$ where $\tilde{f} : X \to \mathbb{R}^2$ denotes the corresponding real map for f,

   ii) $V^J = T^{1,0}X$.

e) Given a holomorphic atlas on X, show that there exists an almost complex structure J on X such that V^J is integrable, that is $[\Gamma(V^J), \Gamma(V^J)] \subset \Gamma(V^J)$ where $[\cdot, \cdot]$ denotes the Lie bracket for vector fields.

Remark: The converse of e) is also true by a theorem of Newlander-Nirenberg. (See for example 'Partial Differential Equations in Several Complex Variables' by So-Chin Chen and Mei-Chi Shaw.)

6. Problem

Let X be a real manifold of dimension $\dim_{\mathbb{R}} X = 2$.

a) Show that any complex subbundle of $\mathbb{C} \otimes TX$ is integrable.

b) Assume that X is orientable.

  i) Show that there exists an almost complex structure J on X. (*Hint: Use that there exists a non vanishing real 2-form*)

  ii) Use the theorem of Newlander-Nirenberg (Remark Problem 4) and a) to prove that X admits a holomorphic atlas.

c) Prove that X admits a holomorphic atlas if and only if X is orientable. (*Hint: Show that a holomorphic atlas induces a positive atlas*)

7. Problem

Let $p : E \to X$ be a holomorphic vector bundle of rank $r > 1$ over a compact complex manifold. The group $\mathbb{C}^*$ acts by scalar multiplication in the fibres on the total space of E, and the action is free on the complement $E^\times$ of the zero section.
(a) Prove that the quotient space $\mathbb{P}(E) = E^\times / C^*$, called the projectivization of E, is a compact complex manifold, and that the projection $p : E \to X$ factors through the quotient map. The induced map $\pi : \mathbb{P}(E) \to X$ is a fibre bundle projection whose fibres are projective spaces.
(b) Show that if L is a line bundle over X, then $\mathbb{P}(E \otimes L)$ is biholomorphic to $\mathbb{P}(E)$.
(c) Consider the rank r vector bundle $\pi^* E \to \mathbb{P}(E)$. A point of $\mathbb{P}(E)$ is represented by a point $x \in X$ and a line ℓ_x in the fiber E_x. Let $\tau_E \subset \pi^* E$ be the line subbundle whose fibre is the line ℓ_x. The bundle τ_E is called the tautological bundle of E. Prove that the restriction of τ_E to a fibre of $\mathbb{P}(E)$ is a tautological bundle over a projective space.
(d) Let $q : L \to X$ be a line bundle. Prove that the tautological bundle of $E \otimes L$ is $\tau_E \otimes q^* L$.
(e) Prove that $E^\times$ and $\tau_E^\times$ are biholomorphic; in other words, the complement of the zero section in E is the same as the complement of the zero section of τ_E. We say that the total space of τ_E is obtained from the total space of E by *blowing up the zero section.*