Complex Geometry - Homework 5

1. Problem

Let $F: \mathbb{C}^n \times \mathbb{R}^m \to \mathbb{C}$ a function that is continuously differentiable in an open neighbourhood of a set $G \times \Gamma$, where G is an open set in $\mathbb{C}^n$ and Γ is a compact set in $\mathbb{R}^m$. Suppose that for any $t_0 \in \Gamma$ the function $z \mapsto F(z, t_0)$ is holomorphic in G. Given an arbitrary measure μ on Γ, show that $f: G \to \mathbb{C}$,

$$f(z) = \int_\Gamma F(z,t)\mu(t)$$

is holomorphic.

2. Problem

Let f be a function defined in a neighbourhood of a set $D' \times D_n$, where D' is a domain in $\mathbb{C}^{n-1}$ and $\overline{D_n}$ is a closed, bounded domain in the z_n-plane. Suppose that f is holomorphic in a neighbourhood of $D' \times \partial D_n$ and that for each fixed $z' \in D'$ the function $w \mapsto f(z', w)$ is holomorphic in D_n. Show that f is holomorphic in $D' \times D_n$.

3. Problem

Consider the analytic set $X = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^3 - z_2^4 = 0\}$.

a) Show that X is a topological manifold, actually X is homeomorphic to $\mathbb{C}$.

b) Show that X is not a complex submanifold of $\mathbb{C}^2$.

4. Problem

Let $U \subset \mathbb{C}^n$ be open and connected. Show that for any non-trivial holomorphic function $f: U \to C$ the complement $U \setminus Z(f)$ of the zero set of f is connected and dense in U.

5. Problem

Prove the following:

a) Let A be an analytic subset of a domain G. Then either $A = G$ or A is nowhere dense, i. e. $\overset{\circ}{A}$.

b) A proper analytic subset of a domain $G \subset \mathbb{C}$ is discrete and closed.

c) Let $A \subset G$ be as in a). If $A \neq G$ then A is nowhere separating, i. e. for any subdomain $D \subset G$ we have that $D \setminus A$ is connected.

6. Problem

Let $a \in \mathbb{C}^n$ be a point A_a, B_a two germs of analytic sets at a and $I_1, I_2 \subset \mathcal{O}_a$ two Ideals. Prove the following:

a) $A_a \subset B_a \Rightarrow J(B_a) \subset J(A_a)$

b) $I_1 \subset I_2 \Rightarrow N(I_2) \subset N(I_1)$

c) $A_a = N(J(A_a))$

d) $I_1 \subset J(N(I_1))$, in general not equal

e) $J(A_a \cup B_a) = J(A_a) \cap J(B_a) \supset J(A_a)J(B_a)$

f) $J(A_a \cap B_a) \supset J(A_a) + J(B_a)$

g) $N(I_1 \cdot I_2) = N(I_1 \cap I_2) = N(I_1) \cup N(I_2)$

h) $N(I_1 + I_2) = N(I_1) \cap N(I_2)$