Complex Geometry - Homework 6

1. Problem

Suppose X is a Riemann surface.

(a) For $U \subset X$ open, let $\mathcal{B}(U)$ be the vector space of all bounded holomorphic functions auf U. For $V \subset U$ let $\mathcal{B}(U) \to \mathcal{B}(V)$ be the usual restriction map. Show that $\mathcal{B}$ is a presheaf which satisfies the first sheaf axiom but not the second sheaf axiom.

(b) For $U \subset X$ open, let $\mathcal{F}(U) := \mathcal{O}^*(U)/\exp\mathcal{O}(U)$. Show that $\mathcal{F}$ with the usual restriction maps is a presheaf which does not satisfy the first sheaf axiom.

2. Problem

Let $\mathcal{F}$ be a presheaf on a topological space X. The presheaf $\mathcal{F}$ is said to satisfy the Identity Theorem if the following holds. If $Y \subset X$ is a domain and $f, g \in \mathcal{F}(Y)$ are elements whose germs coincide at a point of Y, then $f = g$.

(a) Show that the sheaf $\mathcal{O}$ of holomorphic functions on a complex manifold satisfies the Identity Theorem. Show through an example that the sheaf $\mathcal{C}$ of continuous functions on a topological space does not satisfy in general the Identity Theorem. Give other examples of sheaves satisfying or not the Identity Theorem.

(b) Let $s, t \in \Gamma(Z, |\mathcal{F}|)$, where $Z \subset X$. Show that the set $\{x \in Z : s(x) = t(x)\}$ is open in Z.

(c) Assume that $|\mathcal{F}|$ is Hausdorff. Show that $\mathcal{F}$ satisfies the Identity Theorem.

(d) Suppose X is locally connected Hausdorff and $\mathcal{F}$ satisfies the Identity Theorem. Show that $|\mathcal{F}|$ is Hausdorff.

3. Problem

Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}^n$. Let H be a hypersurface in $\mathbb{C}^n$, defined by $F = 0$ for some analytic function F. Let $\mathcal{O}_H$ be the sheaf where $\mathcal{O}_H(U)$ is the holomorphic functions on $U \cap H$. Construct a short exact sequence of sheaves $0 \to \mathcal{O} \to \mathcal{O} \to \mathcal{O}_H \to 0$, and prove that it is exact.

4. Problem

Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be a short exact sequence of sheaves of abelian groups. Assume that $\mathcal{F}$ is flabby. Show that the induced sequence $0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$ is exact.