Complex Geometry - Homework 7

1. Problem
Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be three complexes of abelian groups. We denote all the differential maps by d. Let $\phi^n : A^n \to B^n$ and $\psi^n : B^n \to C^n$ be maps such that $\phi d = d\phi$, $\psi d = d\psi$ and the sequence $0 \to A^n \xrightarrow{\alpha^n} B^n \xrightarrow{\beta^n} C^n \to 0$ is exact. Recall that, in class, we "defined" a map $\delta : H^{n-1}(C^\bullet) \to H^n(A^\bullet)$ as follows: Take $[z] \in H^{n-1}(C^\bullet)$ and lift it to $z \in C^{n-1}$ with $dz = 0$. Find $y \in B^{n-1}$ such that $\psi y = z$. Then $\psi dy = 0$, so we can find $x \in A^n$ with $\phi(x) = dy$. Set $\delta[z] = [x]$, where $[x]$ is the class of x in $H^n(A^\bullet)$.
The point of this exercise is to check the many unchecked claims.

1. Show that $dx = 0$, so that we may speak of the class of x in $H^n(A^\bullet)$.
2. Show that the choice of a lift z for $[z]$, and the choice of a preimage y of z, do not effect the class $[x]$ in $H^n(A)$.
3. Show that $H^n(A^\bullet) \stackrel{\phi^\bullet}{\to} H^n(B^\bullet) \stackrel{\psi^\bullet}{\to} H^n(C^\bullet)$ is exact.
4. Show that $H^{n-1}(B^\bullet) \stackrel{\psi^\bullet}{\to} H^{n-1}(C^\bullet) \stackrel{\delta}{\to} H^n(A^\bullet)$ is exact.
5. Show that $H^{n-1}(C^\bullet) \stackrel{\delta}{\to} H^n(A^\bullet) \stackrel{\phi^\bullet}{\to} H^n(B^\bullet)$ is exact.

2. Problem
Suppose $p_1, \dots, p_n$ are distinct points of $\mathbb{C}$ and let $X := \mathbb{C} \setminus \{p_1, \dots, p_n\}$. Prove that $H^1(X, \mathbb{Z}) \cong \mathbb{Z}^n$.
Hint: Construct a covering $U = (U_1, U_2)$ of X such that U_1 and U_2 are connected and simply connected and $U_1 \cap U_2$ has $n + 1$ connected components.

3. Problem
(a) Let X be a manifold, $U \subset X$ open and $V \Subset U$. Show that V meets only a finite number of connected components of U.
(b) Let X be a compact manifold and $\mathcal{U} = (U_i)_{i\in I}$, $\mathcal{V} = (V_i)_{i\in I}$ be two finite open coverings of X such that $V_i \Subset U_i$ for every $i \in I$. Prove that

$$\operatorname{Im}\left(Z^1(\mathcal{U}, \mathbb{C}) \to Z^1(\mathcal{V}, \mathbb{C})\right)$$

is a finite-dimensional vector space.
(c) Let X be a compact Riemann surface. Prove that $H^1(X, \mathbb{C})$ is a finite dimensional vector space.
Hint: Use finite coverings $\mathcal{U} = (U_i)_{i\in I}$, $\mathcal{V} = (V_i)_{i\in I}$ of X with $V_i \Subset U_i$, such that all the U_i and V_i are isomorphic to disks.

4. Problem

a) Let X be a compact Riemann surface. Prove that the map $H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{C})$, induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$, is injective.
(b) Let X be a compact Riemann surface. Show that $H^1(X,\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module.
Hint: Show first, as in Prob. 3 c), that $H^1(X,\mathbb{Z})$ is finitely generated and then use a) to prove that $H^1(X,\mathbb{Z})$ is free.

5. Problem

(a) Show that $\mathcal{U} = (\mathbb{P}^1 \setminus \{\infty\}, \mathbb{P}^1 \setminus \{0\})$ is a Leray covering for the sheaf Ω of holomorphic 1-forms on $\mathbb{P}^1$.
(b) Prove that $H^1(\mathbb{P}^1,\Omega) \cong H^1(\mathbb{P}^1,\mathcal{U}) \cong \mathbb{C}$ and that the cohomology class of

$$\frac{dz}{z} \in \Omega(U_1 \cap U_2) \cong \mathbb{Z}(\mathcal{U},\Omega)$$

is a basis of $H^1(\mathbb{P}^1,\Omega)$.

6. Problem

Let $U \subsetneq V$ be connected open subsets of $\mathbb{C}^2$. Suppose that we have the following property: For any analytic function f on U, the function f has a holomorphic extension to V. The point of this problem is to show that, in this case $H^1(U,\mathcal{O}) \neq 0$.
Let z and w be coordinates on $\mathbb{C}^2$, and let $(a,b) \in V \setminus U$. Define a complex of sheaves on U by

$$0 \to \mathcal{O} \xrightarrow{\binom{z-a}{w-b}} \mathcal{O}^{\oplus 2} \xrightarrow{(\,-(w-b)\;\; z-a\,)} \mathcal{O} \to 0.$$

Here the first map means that we send f to $((x-a)f,(y-b)f)$ and the second map means that we send (g,h) to $-(w-b)g+(z-a)h$.

1. Check that this is a complex, meaning that the composite of the nontrivial maps is zero.
2. Show that this complex is exact, as a complex of sheaves on U.
3. Write down the corresponding long exact sequence, and show that $H^0(U,\mathcal{O}^{\oplus 2}) \to H^0(U,\mathcal{O})$ is *not* surjective. Deduce that $H^1(U,\mathcal{O}) \neq 0$.