# Complex Geometry - Homework 8

## 1. Problem

Let $f : X \to Y$ be a non-constant holomorphic map between two Riemann surfaces.

(a) The set of *ramification points* of $f$ is defined as $R := \{x \in X : \nu(f, x) > 1\}$. Show that the set of ramification points is closed and discrete.

(b) Show that $f$ is injective when restricted to a small neighbourhood of $x$ $\iff \nu(f, x) = 1 \iff f$ gives a biholomorphic map between a neighbourhood of $x \in X$ and a neighbourhood of $f(x) \in Y$.

## 2. Problem

Recall that a map $f : S \to T$ between two locally compact topological spaces $S, T$ is called *proper* if for any compact set $K \subset T$ the preimage $f^{-1}(K)$ is also compact. Note that if $S$ itself is compact then any map $f$ is proper, since $f^{-1}(K)$ is a closed subset of $S$, hence compact.

(a) Show that if $f : S \to T$ is proper then for any $t \in T$ the pre-image $f^{-1}(t)$ is a finite subset of $S$.

(b) Show that if $f : X \to Y$ is a proper holomorphic map between Riemann surfaces then the image $B = f(R)$ is discrete in $Y$. The set $B$ is called the set of *critical values* of $f$.

## 3. Problem

(a) Let $P \in \mathbb{C}[z]$ be a non-constant polynomial. Calculate the multiplicity $\nu(P, \infty)$ at the point at infinity. What is the set of ramification points of $P$?

(b) The same question as in (a) for a non-constant rational function.

(c) Show that if $f$ is meromorphic function on a Riemann surface and $x$ is a pole of $f$, then the order of the pole $x$ equals $\nu(f, x)$.

## 4. Problem

Suppose that $f : X \to Y$ is a proper, non-constant holomorphic map between Riemann surfaces. For each $y \in Y$ we define an integer $d(y)$ by

$$d(y) = \sum_{x \in f^{-1}(y)} \nu(f, x) \,.$$

(a) Show that for $y \notin B$ we have $d(y) = |f^{-1}(y)|$.

(b) Show that the integer $d(y)$ does not depend on $y \in Y$, called the *degree* of the map $f$.

(c) What is the degree of a non-constant polynomial or rational function (as maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$)?

**5. Problem**

Let $X$ be a compact Riemann surface. Prove that if there is a meromorphic function on $X$ having exactly one pole, and that pole has order $1$, then $X$ is biholomorphic to the Riemann sphere.

**6. Problem**

Let $P \in \mathbb{C}[z, w]$ be an irreducible polynomial

$$P(z,w) = w^n + p_{n-1}(z)w^{n-1} + \ldots + p_1(z)w + p_0(z)\,,$$

such that $(\partial P/\partial z, \partial P/\partial w) \neq (0,0)$.
Consider the Riemann surface $X = \{P(z,w) = 0\}$. Show that the projection $\pi$ of $X$ to the $z$-plane is a proper map and calculate its degree.