1. Basic definitions

Recall the def. of a submanifold from Ana 3.

1.1. Definition (submanifold in $\mathbb{R}^N$)

Let $n, N \in \mathbb{N}_0$. A non-empty set $M \subset \mathbb{R}^N$ is called <u>n-dim submanifold of $\mathbb{R}^N$</u> if any point $x \in M$ has an open neighbourhood U in M and an embedding $\psi : D \to \mathbb{R}^N$ where $D \subset \mathbb{R}^n$ is open and $\psi(D) = U$.

Recall that a smooth map $\psi : D \to \mathbb{R}^N$ is called <u>embedding</u>, if (i) f is an immersion (i.e. the derivative $d\psi(x) : \mathbb{R}^n \to \mathbb{R}^N$ is injective for any $x \in D$) (ii) $\psi : D \to (D)$ is a homeomorphism.
Recall also that a map $f : X \to Y$ between two topological space is called <u>homeomorphism</u> if (i) f bijective (ii) f, f^{-1} continuous

Some terminology which will be used also for Riemann surfaces:
ψ : <u>parametrization</u> ; (U, φ), where $\varphi = \psi^{-1}$: <u>chart</u> ; U : <u>coordinate neighbourhood</u>
For two charts (U_1, φ_1), (U_2, φ_2) we define the <u>transition map</u> $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$

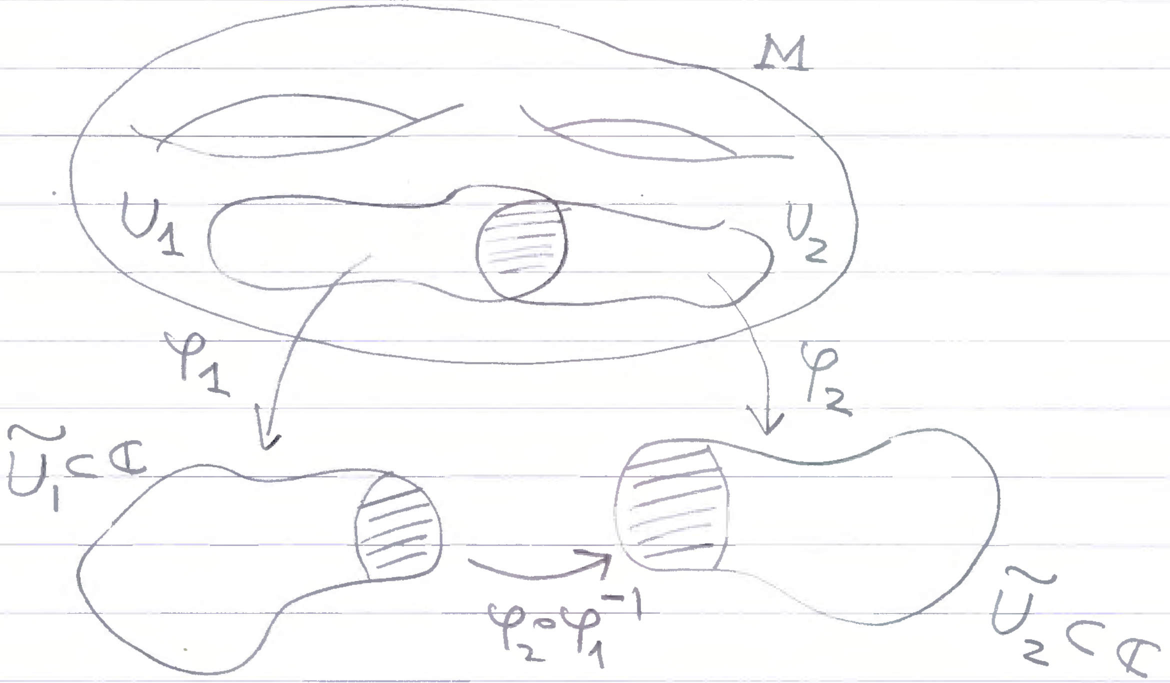
A collection $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is called <u>Atlas</u> if $M = \bigcup_{i \in I} U_i$.

Transition functions are smooth (Lemma 11.4.3 in Skript Ana 3). This observation leads to the definition of (abstract) manifolds, not necessarily as subsets of $\mathbb{R}^N$.

1.2. Definition (n-dim smooth manifold)
Let M be a Hausdorff topological space.
An n-dimensional chart on M is a homeo $\varphi: U \to \tilde{U}$ between an open set $U \subset M$ and an open set $\tilde{U} \subset \mathbb{R}^n$
An atlas on M is a collection $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ of n-dim charts such that for all $i, j \in I$ the transition map $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ is a Diffeomorphism (we say that φ_i, φ_j are compatible)
A maximal atlas is an atlas $\mathcal{A}_{max}$ such that for any atlas $\mathcal{B} \supset \mathcal{A}_{max}$ follows $\mathcal{B} = \mathcal{A}_{max}$
For any atlas $\mathcal{A}$ there exists a maximal atlas $\mathcal{A}_{max} \supset \mathcal{A}$, namely $\mathcal{A}_{max} = \{$all charts compatible with the charts of $\mathcal{A}\}$
Finally, a pair $(M, \mathcal{A}_{max})$ is called n-dim manifold.

Examples: all submanifolds of $\mathbb{R}^N$ are manifolds, but with the abstract definition we can construct a lot of examples of manifolds which are not apriori subsets of $\mathbb{R}^N$, for example the projective space
We concentrate now to the case $n=2$; the charts take values in $\mathbb{R}^2$ which we identify to $\mathbb{C}$ by
$$\mathbb{R}^2 \ni (x,y) \longmapsto x+iy = z \in \mathbb{C}.$$

1.3. Definition (Riemann surface)
A Riemann surface is a connected 2-dim manifold such that the transition maps of an atlas are holomorphic (as maps between open sets in $\mathbb{C}$). A maximal atlas is called a complex or conformal structure.



Note: transition maps are biholomorphic (since $(\varphi_2 \circ \varphi_1^{-1})^{-1} = \varphi_1 \circ \varphi_2^{-1}$ is also a transition map)
thus conformal

1.4. Examples (i) Any domain $D \subset \mathbb{C}$: there is an atlas with only one chart $\{(D, \mathrm{Id}_D)\} = \{(D, z)\}$
The associated maximal atlas is $\{(U, \varphi) : U \subset D$ open, $\varphi : U \to \varphi(U)$ biholomorphic$\}$

(ii) The Riemann sphere can be described in three equivalent ways.

(a) $S^2 = \{(x,y,t) \in \mathbb{R}^3 : x^2+y^2+t^2=1\}$

Use as charts the stereographic projections

$\varphi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$, $\varphi_1(x,y,t) = \dfrac{x+iy}{1-t}$ (proj from North pole)

$\varphi_2 : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$, $\varphi_2(x,y,t) = \dfrac{x-iy}{1+t}$

(proj. from South pole conjugated)

We have $\varphi_1^{-1} : \mathbb{C} \to S^2 \setminus \{(0,0,1)\}$,

$$\varphi_1^{-1}(z) = \frac{(2\operatorname{Re} z, 2\operatorname{Im} z, |z|^2-1)}{1+|z|^2}$$

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}, \quad \varphi_2\circ\varphi_1^{-1}(z) = \frac{\frac{2\bar z}{1+|z|^2}}{\frac{2|z|^2}{1+|z|^2}}$$

$$= \frac{\bar z}{z \cdot \bar z} = \frac{1}{z}$$

(b) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, one point compactification of $\mathbb{C}$

The topology of $\hat{\mathbb{C}}$ is defined as follows: $U \subset \hat{\mathbb{C}}$ is open $\Leftrightarrow$ $U \cap \mathbb{C}$ is open in $\mathbb{C}$ and if $\infty \in U$ $\exists R>0 : \mathbb{C}\setminus\overline{B_R(0)} \subset U$ (i.e. neighbourhood base of ∞ in $\hat{\mathbb{C}}$ is given by complements of closed balls).

Consider the charts $(\mathbb{C}, z)$, $(\hat{\mathbb{C}}\setminus\{0\}, \frac{1}{z})$ where $1/\infty = 0$. The transition map is $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto \frac{1}{z}$.

(c) Projective line $\mathbb{P}^1(\mathbb{C})$. On $\mathbb{C}^2\setminus\{0\}$ consider the equivalence relation $z \sim w \Leftrightarrow \exists \lambda \in \mathbb{C}^* : z = \lambda w$

Set $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) = \mathbb{C}^2\setminus\{0\}/\sim$. The equivalence class of a point $z \in \mathbb{C}^2\setminus\{0\}$ is denoted by $[z] = \{\lambda z : \lambda \in \mathbb{C}^*\}$ which is the complex line through 0 and z without 0. The map $\pi : \mathbb{C}^2\setminus\{0\} \to \mathbb{P}^1$, $\pi(z) = [z]$ is called the projection.