We introduce the quotient topology on $\mathbb{P}^1$: $U \subset \mathbb{P}^1$ is open by definition, if $\pi^{-1}(U)$ is open in $\mathbb{C}^2 \setminus \{0\}$. With this topology $\mathbb{P}^1$ is Hausdorff and compact and π is a continuous map. We define two charts as follows:

$$U_0 = \{ [z_0, z_1] \in \mathbb{P}^1 : z_0 \neq 0 \}, \ \varphi_0 : U_0 \longrightarrow \mathbb{C}, \ \varphi_0([z_0, z_1]) = \frac{z_1}{z_0}$$

$$U_1 = \{ [z_0, z_1] \in \mathbb{P}^1 : z_1 \neq 0 \}, \ \varphi_1 : U_1 \longrightarrow \mathbb{C}, \ \varphi_1([z_0, z_1]) = \frac{z_0}{z_1}$$

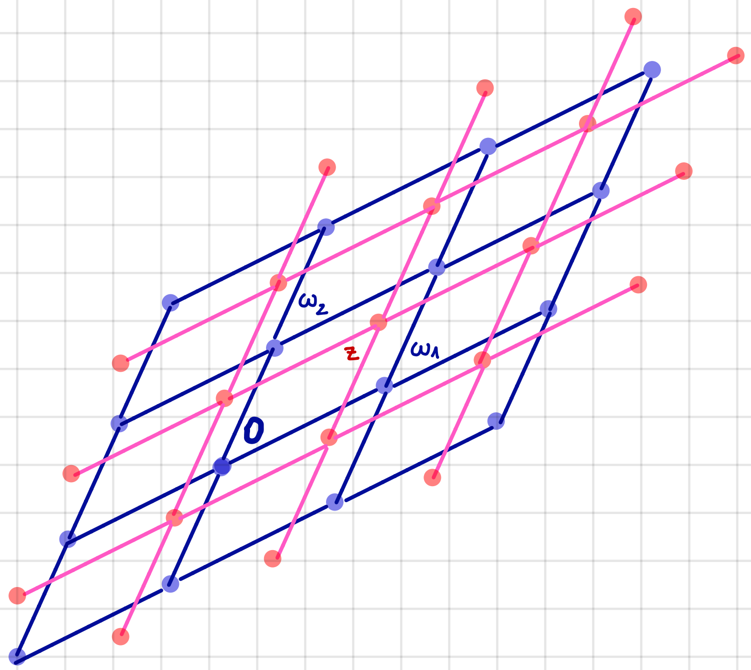
One checks that φ_0 and φ_1 are homeomorphisms and $\varphi_0^{-1}(z) = [1, z]$, $\varphi_1^{-1}(z) = [z, 1]$, thus $\varphi_1 \circ \varphi_0^{-1}, \varphi_0 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*$ are equal and given by the biholomorphic map $z \longmapsto \frac{1}{z}$. Therefore $\mathbb{P}^1$ is a Riemann surface endowed with the atlas $\{(U_0, \varphi_0), (U_1, \varphi_1)\}$.

(iii) Tori Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, equivalently $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0$. This means geometrically that the vectors $\overrightarrow{O\omega_1}, \overrightarrow{O\omega_2}$ are not colinear. The subgroup Γ of $(\mathbb{C}, +)$ generated by ω_1 and ω_2 is called the lattice spanned by ω_1, ω_2 and has the form

$$\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{ m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z} \}.$$

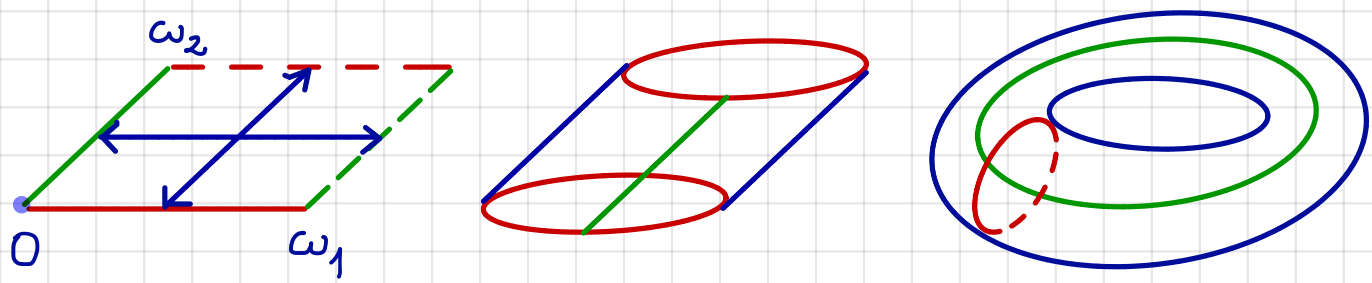
Two complex numbers $z, w \in \mathbb{C}$ are called equivalent modulo Γ if $z - w \in \Gamma$ that is if there exists $\gamma \in \Gamma$ with $z = w + \gamma$, so that z is obtained from w by a translation with $\gamma \in \Gamma$. The set of all equivalence classes is denoted by $\mathbb{C}/\Gamma$. The equivalence class of z is denoted $[z]$ and the map $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$, $\pi(z) = [z]$ is called the canonical projection.

The set $\mathbb{C}/\Gamma$ has a nice geometric realization as a parallelogram $P = \{ z \in \mathbb{C} : z = a\omega_1 + b\omega_2, \ a, b \in [0, 1) \}$. Every point in $\mathbb{C}$ is equivalent to a point in P and no points of P are equivalent. Actually, every $z \in \mathbb{C}$ can be written uniquely as $z = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{R}$ and $z \sim z' := \{a\}\omega_1 + \{b\}\omega_2 \in P$, where $\{x\} \in [0,1)$ is the fractional part of $x \in \mathbb{R}$. Moreover, z' is the unique point from $[z]$ in P, $[z] \cap P = \{z'\}$. Thus $\pi : P \to \mathbb{C}/\Gamma$ is bijective.



Lattice points and the equivalence class $[z] = z + \Gamma$ of a point z.

Since points on the opposite edges of the parallelogram P are are equivalent, $z \sim z + \omega_1$, $z \sim z + \omega_2$, we can glue these pair of edges together and obtain a torus.

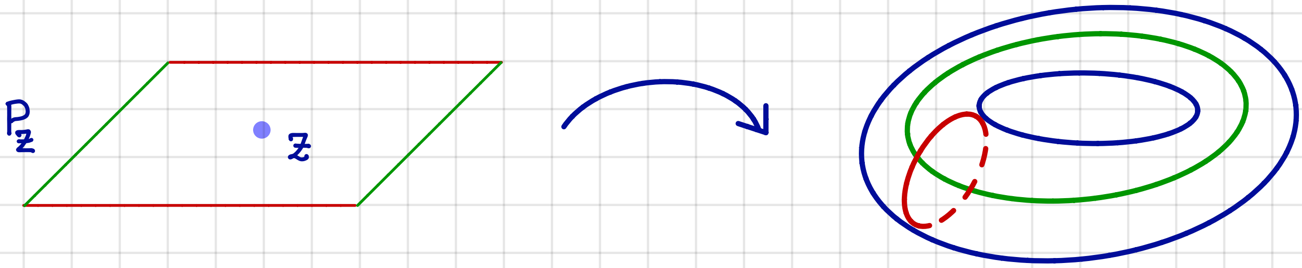


We introduce the quotient topology on $\mathbb{C}/\Gamma$:

$$U \subset \mathbb{C}/\Gamma \text{ is open } :\Longleftrightarrow \pi^{-1}(U) \text{ is open in } \mathbb{C}.$$

One checks that this is indeed a topology and π is then continuous. Since $\mathbb{C}$ is connected, then $\mathbb{C}/\Gamma = \pi(\mathbb{C})$ is connected. Since $\overline{P}$ is compact also $\mathbb{C}/\Gamma = \pi(\overline{P})$ is compact. The projection π is an open mapping, that is, maps open sets to open sets. Indeed, if $V \subset \mathbb{C}$ is open, then $\pi^{-1}(\pi(V)) = \bigcup_{\gamma \in \Gamma} (\gamma + V)$ is open, so $\pi(V)$ is open.

We define an atlas on $\mathbb{C}/\Gamma$ in the following way. For any $z \in \mathbb{C}$ let $P_z = \{z + a\omega_1 + b\omega_2 : a, b \in (-\frac{1}{2}, \frac{1}{2})\}$ be the open parallelogram centered at z. Then $\pi|_{P_z} : P_z \to \pi(P_z)$ is a homeomorphism: it is bijective, continuous and $\varphi_z := (\pi|_{P_z})^{-1}$ is continuous, since π is open.



It is clear that the sets $\{\pi(P_z), z \in \mathbb{C}\}$ form an open covering of $\mathbb{C}/\Gamma$. Let $z_1, z_2 \in \mathbb{C}$, $\varphi_1 = \varphi_{z_1}$, $\varphi_2 = \varphi_{z_2}$. Let $U = \pi(P_{z_1}) \cap \pi(P_{z_2})$ and let $f = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U) \to \varphi_2(U)$. Thus $f(z) = (\pi|_{P_{z_2}})^{-1} \pi|_{P_{z_1}}(z)$ so $\pi(f(z)) = \pi(z)$, that is $f(z) \sim z$, $f(z) - z = \omega(z) \in \Gamma$.

Since f is continuous, also ω is continuous. Now Γ is a discrete set, thus ω is a locally constant function, so holomorphic.

The transition functions being holomorphic, it follows that $\{(\pi(P_z), \varphi_z) : z \in \mathbb{C}\}$ is a holomorphic atlas on $\mathbb{C}/\Gamma$, which becomes a Riemann surface.

1.5. Definition (holomorphic map)
Let X, Y be Riemann surfaces; a continuous map $f: X \to Y$ is called *holomorphic* if for every pair of charts (U, φ) on X, (V, ψ) on Y, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \tilde{V}$$

is holomorphic. A map $f: X \to Y$ is called *biholomorphic* if f is bijective and f, f^{-1} are holomorphic.
A holomorphic map $f: X \to \mathbb{C}$ is called *holomorphic function*. Notation: $\mathcal{O}(X) = \{f: X \to \mathbb{C} : f \text{ holom}\}$

1.6. Remarks (i) If X, Y, Z are RS and $f: X \to Y$, $g: Y \to Z$ holom, then the composition $g \circ f: X \to Z$ is holom.
(ii) The condition in the def 1.5 has to be verified only for charts in *some* atlases on X, Y. Due to the compatibility it is automatically verified for all charts in the maximal atlases on X, Y.
(iii) The sum and product of holom fct is holom, constant fct are holom. Thus $\mathcal{O}(X)$ is a $\mathbb{C}$-algebra.
(iv) A chart $\varphi: U \to \tilde{U} \subset \mathbb{C}$ is holom: this is just the compatibility with other charts.
(v) Riemann's removable singularities theorem (Riemannscher Hebbarkeitssatz) is still valid on RS: Let X be a Riemann surface, $a \in X$, $f \in \mathcal{O}(X \setminus \{a\})$ bounded in some neighborhood of a. Then f can be extended to a function $\tilde{f} \in \mathcal{O}(X)$.

1.7. Examples (i) Let $P \in \mathbb{C}[z]$ be a non-constant polynomial. Since $\lim_{z \to \infty} P(z) = \infty$ so we can extend P by continuity to $P: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This map is holomorphic.