(ii) S^2, $\hat{\mathbb{C}}$, $\mathbb{P}^1$ are biholomorphic: define $f: S^2 \to \hat{\mathbb{C}}$

$$f(x,y,t) = \begin{cases} \varphi_1(x,y,t), & (x,y,t) \neq (0,0,1) \\ \infty, & (x,y,t) = (0,0,1) \end{cases}$$ where φ_1 is the stereographic projection. Then f is biholomorphic.

Define also $g: \mathbb{P}^1 \to \hat{\mathbb{C}}$, $g([z_0, z_1]) = \begin{cases} z_1/z_0 & z_0 \neq 0 \\ \infty & z_0 = 0 \end{cases}$

Then g is biholom.

1.8. Identity theorem Let X, Y be RS and $f_1, f_2: X \to Y$ holom. Suppose that the set $A = \{x \in X : f_1(x) = f_2(x)\}$ has a limit point $a \in X$. Then $f_1 = f_2$.

Proof Set $G := \{x \in X : \exists W$ open neighb of x s.t. $f_1|_W = f_2|_W\}$

We show that $G \neq \emptyset$, G open and closed.

Let a be a limit point of A. We show that $a \in G$.

First note that $a \in A$ i.e. $f_1(a) = f_2(a)$. Indeed, $F = (f_1, f_2): X \to Y \times Y$ is continuous and $\Delta_Y = \{(y,y) \in Y \times Y : y \in Y\}$ is closed, since Y is Hausdorff. Thus $A = F^{-1}(\Delta_Y)$ closed.

a limit point $:\Leftrightarrow$ $\forall U$ neighb of a, $U \cap A$ infinite

Especially $a \in \bar{A} = A$.

Consider now a chart (U, φ) around a w/ U connected, (V, ψ) around $f_1(a) = f_2(a)$ s.t. $f_i(U) \subset V$, $i = 1,2$.

Hence $g_i = \psi \circ f_i \circ \varphi^{-1}: \tilde{U} \to \tilde{V}$ holom and $g_1 = g_2$ on $\varphi(A)$, which has the limit point $\varphi(a) \in \tilde{U}$

Identity theorem for holom fcts in $\mathbb{C}$ $\rightsquigarrow$ $g_1 = g_2$ on $\tilde{U}$
$\rightsquigarrow$ $f_1 = f_2$ on U $\rightsquigarrow$ $U \subset A$ $\rightsquigarrow$ $a \in G$

G is closed: if x is a limit point of $G \subset A$, then as above follows $x \in G$

G is open by definition

Since X is connected $\rightsquigarrow$ $G = X$ $\rightsquigarrow$ $A = X$ i.e. $f_1 = f_2$ on X. ▨

1.9 Corollary (Discretness of fibers)

Let $f: X \to Y$ holom, non-const. Then for any $b \in Y$ the fiber $f^{-1}(b)$ is discrete and closed

(Recall that $A \subset X$ is discrete $:\Leftrightarrow \forall a \in A$ is isolated in A i.e. $\exists U$ open in $X : U \cap A = \{a\}$ $\Leftrightarrow$ A has no limit points in X, which belong to A)

1.10 Remark (a technical topological point)

A topological space is called second countable (or has a countable base) if there exists a countable family of open sets $\{U_j\}_j$ s.t. every open set is a union of some of the U_j.

Example: a countable base of $\mathbb{R}^n$ is the collection of balls with rational radius and center with rational components. Subsets of $\mathbb{R}^n$ are second countable

Theorem Let X be a topological manifold with countable base. Then the topology of X is given by a metric. Especially:

$K \subset X$ compact $\Leftrightarrow$ K sequentially compact

$f: X \to Y$ continuous $\Leftrightarrow$ f sequentially continuous

This follows from the metrization thm of Urysohn: X regular, Hausdorff and second countable $\Rightarrow$ X metrizable (see Kelley, General Topology). A manifold is locally compact and Hausdorff hence regular i.e. one can separate by open neighb a point and a closed set not containing it.

Theorem (Radó, 1924) A Riemann surface is second countable (see Forster §23)

The theorem is non-trivial only for non-compact surfaces. A compact surface can be covered by a finite number of discs and these are second countable.

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