1.11 Proposition Let $f \in \mathcal{M}(X)$ and set $f(p) = \infty$ for $p \in P(f)$. Then $f: X \to \hat{\mathbb{C}}$ is a holom. map. Conversely, if $f: X \to \hat{\mathbb{C}}$ is holom, $f \not\equiv \infty$, then $f^{-1}(\infty)$ is discrete and $f: X \setminus f^{-1}(\infty) \to \mathbb{C}$ is holom. ▤

We identify thus

$$\mathcal{M}(D) \longleftrightarrow \{ f: X \to \hat{\mathbb{C}} \text{ holom} : f \not\equiv \infty \}$$

2. Elementary properties of holomorphic maps

Recall from Complex Analysis that we defined the order of $f \in \mathcal{O}(D)$, $D \subset \mathbb{C}$, in $z_0 \in D$ by

$$\operatorname{ord}_{z_0}(f) = \begin{cases} \min\{n \in \mathbb{N}_0 : f^{(n)}(z_0) \neq 0\}, & \text{if } f \not\equiv 0 \text{ in a neighb of } z_0 \\ \infty, & \text{otherwise} \end{cases}$$

The multiplicity with which the value $f(z)$ is taken is $\nu(f, z_0) := \operatorname{ord}_{z_0}(f - f(z_0))$.

We proved that:

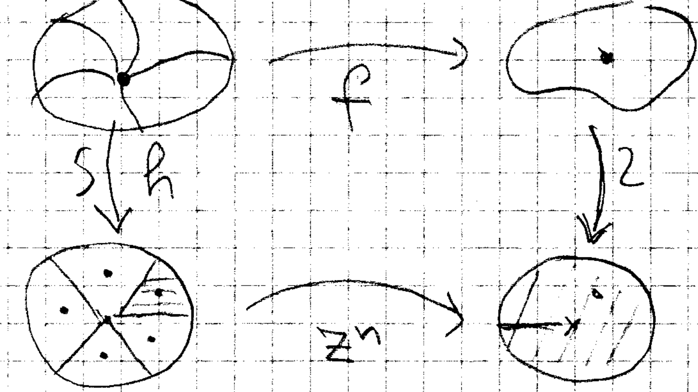
$$\nu(f, z_0) = n < \infty \iff \exists \hat{f} \in \mathcal{O}(D),\ \hat{f}(z_0) \neq 0,\ \forall z \in D: \ f(z) = f(z_0) + (z - z_0)^n \hat{f}(z)$$

2.1. Theorem (local form in $\mathbb{C}$) Let $f \in \mathcal{O}(D)$ non-const and $z_0 \in D$. Then there exists a ball $B = B_r(z_0)$ and a biholom map $h: B \to h(B)$ s.t.

$$f(z) = f(z_0) + h^n(z), \quad z \in B$$

where $n = \nu(f, z_0)$. Especially $(f \circ h^{-1})(w) - f(z_0) = w^n$

Proof Write $f(z) = f(z_0) + (z - z_0)^n g(z)$, $g(z_0) \neq 0$, $n = \nu(f, z_0)$, $1 \le n < \infty$. Choose B s.t. $g|_B \neq 0$. Existence of holomorphic roots $\to \exists s \in \mathcal{O}(B)$ $s^n = g|_B$. Set $h := (z - z_0)s \leadsto h^n = (z - z_0)^n s^n = (z - z_0)^n g = f - f(z_0)$. Moreover, $s^n(z_0) = g(z_0) \neq 0$ $\leadsto s(z_0) \neq 0$. But $h'(z) = s(z) + (z - z_0)s'(z) \to h'(z_0) = s(z_0) \neq 0$ $\leadsto h$ biholom at $z_0 \leadsto$ conclusion ▤



Each sector of angle $\frac{2\pi}{n}$ is mapped biholomorphically on the disk without a radius.
Each point in the image different from zero has exactly n preimages.

2.2 Theorem $f: X \to Y$ holom, non-const; $a \in X$, $b = f(a)$.
Then there exists $n \in \mathbb{N}$, charts (U, φ) around a, (V, ψ) around b s.t. $\varphi(a) = 0 = \psi(b)$, $\varphi(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}(z) = z^n$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ B_\varepsilon(0) = \tilde{U} \arrow[r, "z^n"'] & \tilde{V} = B_{\varepsilon^n}(0) \end{tikzcd}$$

The number n is called multiplicity of f at a

Example $P \in \mathbb{C}[z]$, grad $P = n \rightsquigarrow P: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $P(\infty) = \infty$
$(U, \varphi) = (V, \psi) = (\hat{\mathbb{C}} \setminus \{0\}, \frac{1}{z}) \rightsquigarrow \psi \circ P \circ \varphi^{-1}(z) = \dfrac{1}{P(1/z)}$.
$= \dfrac{1}{a_n(\frac{1}{z})^n + \dots + a_0} = \dfrac{z^n}{a_n + a_{n-1}z + \dots + a_0 z^n}$ has a zero of order n at $0 \rightsquigarrow \nu(P, \infty) = n$. ∎

2.3 Open mapping theorem $f: X \to Y$ holom, non-const $\rightsquigarrow$ f is open

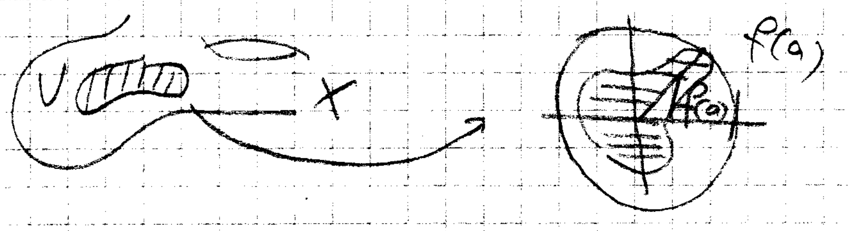
Proof Let $D \subset X$ open; for any $a \in D$, $\exists (U, \varphi)$, $U \subset D$ and (V, ψ) as in Thm 2.2 $\rightsquigarrow f(U) = V \rightsquigarrow f(D)$ neighb of $f(a) \rightsquigarrow f(D)$ open.

2.4 Theorem $f: X \to Y$ holom, inj $\rightsquigarrow f: X \to f(X)$ biholomorph

Proof f inj $\rightsquigarrow \nu(f, z) = 1 \rightsquigarrow \psi \circ f \circ \varphi^{-1}(z) = z \rightsquigarrow$
$\rightsquigarrow \varphi \circ f^{-1} \circ \psi^{-1}(z) = z \rightsquigarrow f^{-1}$ holom ∎

2.5 Maximum principle $f: X \to Y$ holom, non-const
$\rightsquigarrow$ the modulus $|f|$ has no local maximum

Proof By contradiction: assume $|f(a)| = \max_U |f|$, U open
$\rightsquigarrow f(U) \subset \overline{B_{|f(a)|}(0)}$. But $f(U)$ open, thus



$f(U) \subset \overset{\circ}{\overline{B_{|f(a)|}(0)}}$
$= B_{|f(a)|}(0)$
$\rightsquigarrow f(a) \in B_{|f(a)|}(0)$
$\rightsquigarrow |f(a)| < |f(a)|$ ↯

2.6 Theorem
$f: X \to Y$ holom, non-const, X cpt $\rightsquigarrow$ Y cpt and f surj

Proof $f(X)$ open; $f(X)$ cpt $\rightsquigarrow f(X)$ closed $\rightsquigarrow f(X) = Y$.

2.7 Corollary Every holom fct on a cpt RS is const

2.8 Theorem (Liouville) $f \in \mathcal{O}(\mathbb{C})$ bounded $\rightsquigarrow f$ const

Proof Riemann's removable sing thm $\rightsquigarrow \hat{f}: \hat{\mathbb{C}} \to \mathbb{C}$ holom $\rightsquigarrow \hat{f}$ so f const ▨

2.9 Theorem $P \in \mathbb{C}[z]$, grad $P = n \rightsquigarrow P: \mathbb{C} \to \mathbb{C}$ surj
More precisely, for $w \in \mathbb{C}$, $P'(w) \neq 0$ we have
$\# P^{-1}(w) = n$

Proof $P: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $P(\infty) = \infty$ is holom $\rightsquigarrow P$ surj

2.10 Theorem (Liouville) Let Γ be a lattice in $\mathbb{C}$ and let $f \in \mathcal{O}(\mathbb{C})$ be periodic with respect to Γ i.e. $f(z+\gamma) = f(z)$ for all $z \in \mathbb{C}$, $\gamma \in \Gamma$ (f is called doubly periodic).
Then f is constant.

Proof Define $\tilde{f}: \mathbb{C}/\Gamma \to \mathbb{C}$, $\tilde{f}(z+\Gamma) = f(z)$ which is well defined and holomorphic ($\tilde{f} \circ \varphi_z^{-1} = \tilde{f} \circ (\pi|_{P_z}) = f|_{P_z} \in \mathcal{O}(P_z)$).
Hence $\tilde{f}$ is constant, thus also f ▨