3. Branched and unbranched coverings

Let us consider a polynomial $P \in \mathbb{C}[z]$, $\deg P = n \geq 1$. Let $C = \{z \in \mathbb{C} : P'(z) = 0\}$, which is a finite set (with at most $n-1$ elements). If $w \in \mathbb{C} \setminus P(C)$, then $P^{-1}(w) = \{z_1, \dots, z_n\}$ has exactly n elements, and for each j there exist neighbourhoods V_j of z_j and U_j of w such that $P|_{V_j} : V_j \to U_j$ is biholomorphic (since $P'(z_j) \neq 0$). We can in fact take $U_1 = \dots = U_n = U$ and $V_1, \dots, V_n$ mutually disjoint.

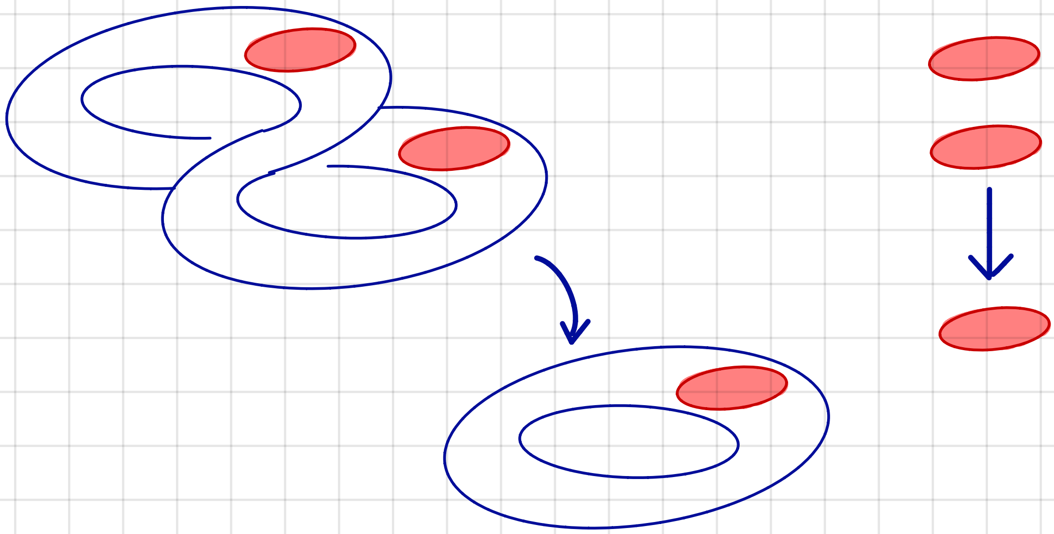
Then for each $w' \in U$, $P^{-1}(w') \cap V_j$ consists of one element z'_j and for $w' \to w$ we have $P^{-1}(w') \to P^{-1}(w)$, in the sense that $z'_j \to z_j$ for any $j = 1, \dots, n$.

In the geometric study of continuous maps $p : Y \to X$ between topological surfaces it is convenient to adopt the following point of view: p can be seen as a continuous family of the inverse images $p^{-1}(x)$ parametrized by $x \in X$. In this context we call X the basis, Y the total space and $p^{-1}(x)$ the fiber over x.

The map p is called discrete (resp. finite) if each fiber is discrete (resp. finite).

The theory of coverings is purely topological, so one can work only with topological spaces. For some results supplementary hypotheses are needed, so we will work from the beginning with spaces having all necessary properties, namely with topological manifolds.

3.1. Definition Let X be a topological manifold. A *covering* of X is a pair (Y, p) consisting of a topological surface Y, called *covering space*, and a continuous map $p: Y \to X$, called *projection* or *covering map* such that the followig holds:
Each point $x \in X$ has a connected open neighbourhood U such that each connected component of $p^{-1}(U)$ is mapped homeomorphically on U by p. We say that U is *evenly covered* by p.



3.2 Lemma Every covering map is a local homeomorphism and and open map. An injective covering map is a homeomorphism.

3.3. Examples (1) $p: \mathbb{R} \to S^1$, $p(t) = \exp(it)$. For $z_0 = e^{it_0}$ the neighbourhood $U = \{e^{it} : t \in (t_0 - \pi, t_0 + \pi)\} = S^1 \setminus \{-e^{it}\}$ is evenly covered, $p^{-1}(U) = (t_0 - \pi, t_0 + \pi) + 2\pi\mathbb{Z}$.

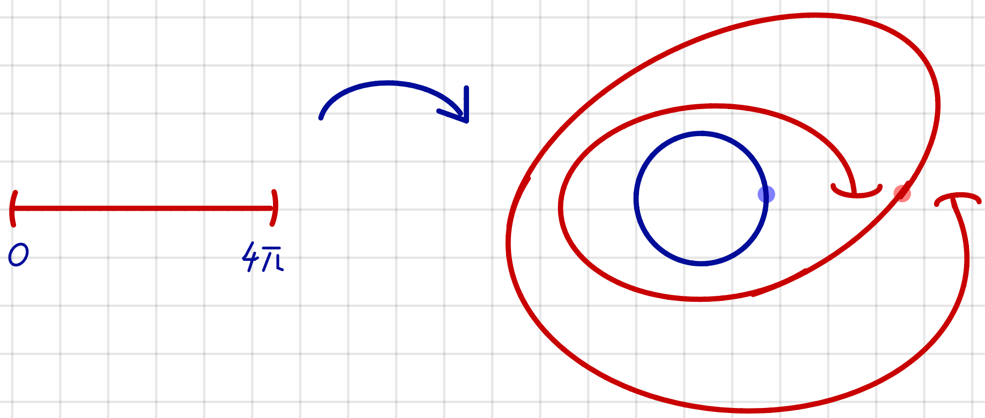
(2) $\exp: \mathbb{C} \to \mathbb{C}^*$ is a covering map. For $z_0 = |z_0| e^{i \arg z_0}$ the neighbourhood $U = \mathbb{C} \setminus \mathbb{R}_- e^{i \arg z_0}$ is evenly covered:

$$\exp^{-1}(U) = \{z \in \mathbb{C} : \operatorname{Im} z \in (\arg z_0 - \pi, \arg z_0 + \pi)\} + 2\pi\mathbb{Z}$$

(3) $p : \mathbb{C}^* \to \mathbb{C}^*$, $p(z) = z^n$ is a covering map. Let $z_0 \in \mathbb{C}^*$ and choose $w_0 \in \mathbb{C}^*$ an n-th rooth of z_0, $w_0^n = z_0$. The map p is a local homeomorphism ($p'(z) \neq 0$, $z \in \mathbb{C}^*$) thus there are open neighbourhoods V_0 of w_0 and U of z_0 such that $p|_{V_0} : V_0 \to U$ is a homeomorphism. Then $p^{-1}(U) = V_0 \cup \zeta V_0 \cup \dots \zeta^{n-1} V_0$, where ζ is a primitive root of unity (e.g. $\zeta = \exp(2\pi i/n)$) and U is evenly covered.

(4) Suppose $\Gamma \subset \mathbb{C}$ is a lattice and let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ be the canonical quotient map. Let $z \in \mathbb{C}$. Then $U = \pi(P_z)$ is evenly covered since $\pi^{-1}(U) = \pi^{-1}(\pi(P_z)) = P_z + \Gamma$ and for each $\gamma \in \Gamma$, $\pi : P_z + \gamma \to U$ is a homeomorphism.

(5) A surjective local homeomorphism may not be a covering map. Let $Y = (0, 4\pi)$ und $p : Y \to S^1$, $p(t) = e^{it}$. Then p is a local homeomorphism (because it is the restriction of the covering $\mathbb{R} \to S^1$) and is surjective. However, the point $1 \in S^1$ has no evenly covered neighbourhood.



We want to find conditions when a local homeomorphism is a covering map.

<u>3.4 Definition</u> Let X, Y be topological manifolds. A continuous map $p: Y \to X$ is called <u>proper</u> if for any compact set $K \subset X$ the preimage $p^{-1}(K)$ is compact.

<u>Example</u> If $p: Y \to X$ is continuous and Y is compact, then p is proper.

<u>3.5 Properties of proper maps</u> If p is proper, then

(1) p is closed, i.e. the image of any closed set is closed.

(2) If $x \in X$ and V is a neighbourhood of $p^{-1}(x)$, there exist a neighbourhood U of x such that $p^{-1}(U) \subset V$.

If p is proper and discrete, then

(3) p is finite, i.e. $p^{-1}(x)$ is finite for every $x \in X$.

<u>Proof</u> (1) In a locally compact space a subset is closed precisely when its intersection with every compact set is compact.

(2) We may assume that V is open and thus $Y \setminus V$ is closed. Then $p(V \setminus Y)$ is closed and $x \notin p(V \setminus Y)$. Thus $U = X \setminus p(Y \setminus V)$ is an open neighbourhood of x such that $p^{-1}(U) \subset V$.

(3) $p^{-1}(x)$ is a compact discrete set of Y. $\blacksquare$

<u>3.6 Theorem</u> Let X, Y be topological surfaces and $p: Y \to X$ be a proper local homeomorphism. Then p is a covering map.

Proof Suppose $x \in X$ is arbitrary and let $p^{-1}(x) = \{y_1, \dots, y_n\}$. Since p is a local homeomorphism for every $j = 1, \dots, n$ there exists an open neighbourhood U_j of x and an open neighbourhood W_j of y_j such that $p|_{W_j} : W_j \to U_j$ is a homeomorphism. We may assume that the W_j are pairwise disjoint. Now $W_1 \cup \dots \cup W_n$ is a neighbourhood of $p^{-1}(x)$. Thus by 3.5 (2) there exists an open connected neighbourhood $U \subset U_1 \cap \dots \cap U_n$ of x such that $p^{-1}(U) \subset W_1 \cup \dots \cup W_n$.
If we let $V_j = W_j \cap p^{-1}(U)$ then the V_j are disjoint open sets with $p^{-1}(U) = V_1 \cup \dots \cup V_n$, all the maps $p_j|_{V_j} : V_j \to U$ are homeomorphisms and $y_j \in V_j$, for $j = 1, \dots, n$. $\blacksquare$

3.7 Theorem Let $p : Y \to X$ be a covering. Assume tha X is connected. Then the cardinality of $p^{-1}(x)$ is the same for all $x \in X$. In particular, if Y is non-empty, p is surjective.

Proof Let $x \in X$ arbitrary and let U be an open neighbourhood of x which is evenly covered by p. Let $p^{-1}(U) = \bigcup_{j \in J} V_j$ be the disjoint union of the components of $p^{-1}(U)$. For any $x' \in U$ we have $p^{-1}(x') \subset p^{-1}(U)$ and $p^{-1}(x') \cap V_j$ has one element y_j. There is thus a bijection $J \to p^{-1}(x')$, $j \mapsto y_j$, so each fiber $p^{-1}(x')$, for $x' \in U$ has the same cardinality as J. Thus the cardinality function $Y \to \mathbb{N}_0 \cup \{\infty\}$, $x \mapsto |p^{-1}(x)|$ is locally constant.
Since X is connected, the function is constant. Indeed, consider for any $c \in \mathbb{N} \cup \{\infty\}$ the set $A_c = \{x \in X : |p^{-1}(x)| = c\}$.

This set is open from the above argument. It is also closed since

$A_c = X \setminus \bigcup_{d \neq c} A_d$. Thus, if $A_c \neq \emptyset$, then $A_c = X$. $\blacksquare$

The number $|p^{-1}(x)|$ is called the number of sheets of the covering.