For the study of Riemann surfaces it is necessary to consider also branched coverings.

3.8 Definition Let X be a topological surface. A branched covering is a pair (Y,p) consisting of a topological surface Y, called covering space, and a continuous map $p: Y \to X$, called projection or covering map such that the followig holds:

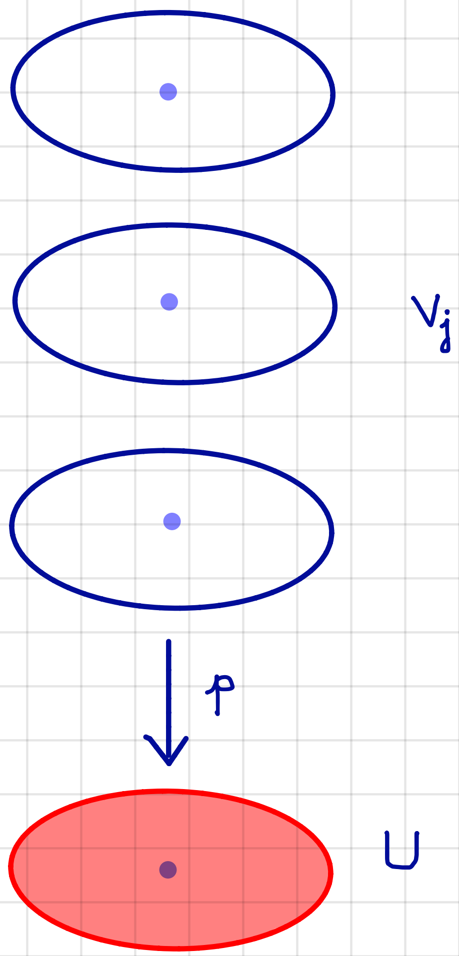
Each point $x \in X$ has a connected open neighbourhood U such that if $p^{-1}(U) = \bigcup_{j\in J} V_j$ is the decomposition of $p^{-1}(U)$ in connected components, for each $j \in J$, $p^{-1}(x) \cap V_j$ consists of one element $y_j \in p^{-1}(x)$ and there exists charts $\varphi_j: U \to \mathbb{C}$, $\psi_j: V_j \to \mathbb{C}$ with $\varphi_j(x) = 0 = \psi_j(y_j)$, and $p(V_j) = U$, $\varphi_j p \psi_j^{-1}: \psi_j(V_j) \to \varphi_j(U)$, $z \longmapsto z^{n_j}$ $(n_j \geq 1)$.

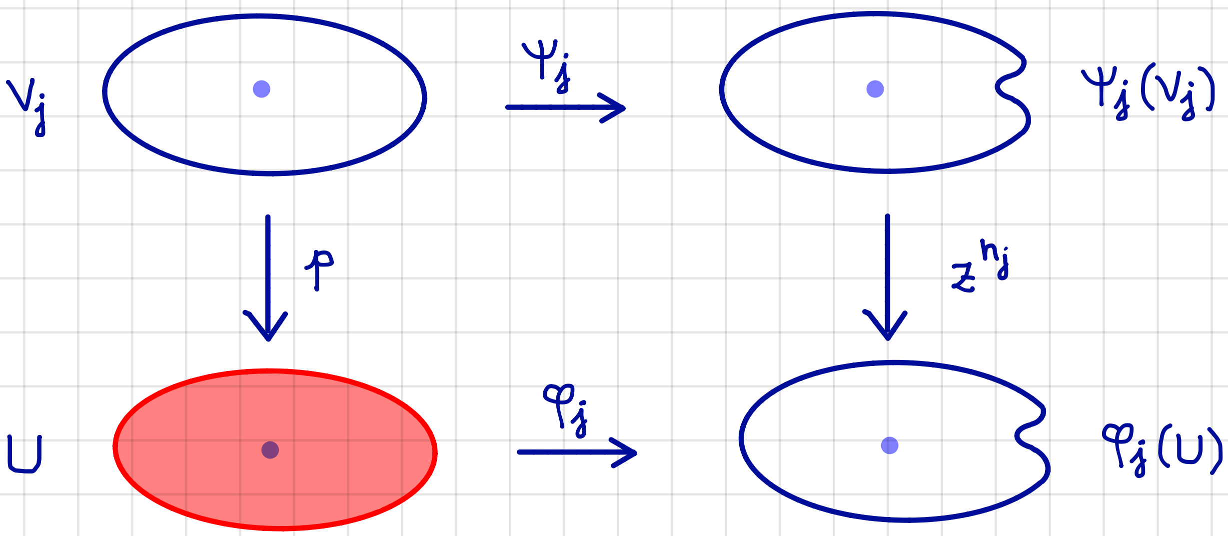
The number $n_j = \nu(p, y_j)$ is called the multiplicity and $r(p,y_j) = n_j - 1$ is called the ramification order of p at y_j. A point $y_j \in p^{-1}(x)$ is called ramification (or critical) point of f if $\nu(p,y_j) \geq 2$ and regular point if $\nu(p,y_j) = 1$ (in this case $p|_{V_j}: V_j \to U$ is a homeomorphism).

A point $x \in X$ is called regular value of p if all points $y \in p^{-1}(x)$ are regular, and critical value if there exists a critical point $y \in p^{-1}(x)$.

3.9 Theorem Let $p: Y \to X$ be a proper holomorphic non-constant map betwenn Riemann surfaces. Then p is a branched covering.

Proof Since p is holomorphic and non-constant, we know that p is surjective and has discrete fibers.


V_j
p
U


V_j
ψ_j
$\psi_j(V_j)$
p
z^{n_j}
U
φ_j
$\varphi_j(U)$

Since p is proper the fibers are finite. Let $x \in X$ and set $p^{-1}(x) = \{y_1, \dots, y_n\}$. We follow the proof of Theorem 3.6 with the necessary modifications. We choose namely charts (U_j, φ_j), (W_j, ψ_j) around x and y_j such that $\varphi_j(x) = 0 = \psi_j(y_j)$, and $p(W_j) = U_j$ and

$$\varphi_j \, p \, \psi_j^{-1} : \psi_j(V_j) \longrightarrow \varphi_j(U_j), \quad z \longmapsto z^{n_j}$$

and conclude as in Theorem 3.6. ▨

<u>3.10 Theorem</u> Let $p : Y \to X$ be a proper branched covering. Then the set $R \subset Y$ of critical points and the set $p(R)$ of critical values are closed and discrete. The map $Y \setminus p^{-1}(p(R)) \xrightarrow{p} X \setminus p(R)$ is a proper unbranched covering.

<u>Proof</u> If $y \in Y$ is a regular point, then p is a homeomorphism in a neighbourhood V of y, so all points of V are regular. Thus the set $Y \setminus R$ of regular points is open, so R is closed. Since p is proper, also $p(R)$ is closed. By the local form of p around a critical point it is clear that R and $p(R)$ are discrete.

The map $Y \setminus p^{-1}(p(R)) \xrightarrow{p} X \setminus p(R)$ is a proper local homeomorphism so by Theorem 3.6 is a covering. ▨

3.11 Theorem Let $p: Y \to X$ be a proper branched covering where X is connected. Then the function $d: X \to \mathbb{N}$, $d(x) = \sum_{y \in p^{-1}(x)} \nu(p,y)$ is constant and it is called the degree of p.

The conclusion means that p takes each value of $x \in X$, counting multiplicities, d times. Note that by definition of branched coverings $p^{-1}(x)$ is discrete so finite, p being proper, see 3.5 (3). Thus the sum defining $d(x)$ is finite so $d(x) \in \mathbb{N}$ indeed.

Proof Let $R \subset Y$ be the set of critical points. By Theorem 3.11 the map $Y \setminus p^{-1}(p(R)) \to X \setminus p(R)$ is an unbranched covering. Being proper it is a finite covering. For $x \in X \setminus p(R)$ and $y \in p^{-1}(x)$ we have $\nu(p,y) = 1$ Thus $d(x) = |p^{-1}(x)|$ is the number of sheets of this covering, say d.

If $x \in p(R)$ consider neighbourhoods U and V_j as in the definition. Then for $x' \in U \setminus \{x\}$ we have $|p^{-1}(x')| = d$ and $|p^{-1}(x') \cap V_j| = n_j = \nu(p,y_j)$, since on V_j looks like $z \mapsto z^{n_j}$. Hence

$$d = d(x') = \sum_{j \in J} |p^{-1}(x') \cap V_j)| = \sum_{y_j \in p^{-1}(x)} \nu(p,y_j) = d(x).$$

$\blacksquare$

3.12 Example A polynomial $P \in \mathbb{C}[z]$ of degree $n > 1$ can be considered as a map $P: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $P(\infty) = \infty$, which is a branched covering of degree $n = \deg(P)$. The critical points are the zeroes of P' and ∞. Note that $P^{-1}(\infty) = \{\infty\}$ and $\nu(P,\infty) = n$.

<u>3.13 Theorem</u> On any compact Riemann surface every nonconstant meromorphic function has as many zeros as poles, where each is counted according to multiplicities.

<u>Proof</u> A meromorphic function is a holomorphic map $f: X \to \hat{\mathbb{C}}$, thus a branched covering of degree d. Then

$$\underbrace{\sum_{x \in f^{-1}(0)} \nu(f,x)}_{\text{Number of zeros}} = d = \underbrace{\sum_{x \in f^{-1}(\infty)} \nu(f,x)}_{\text{Number of poles}}.$$

$\blacksquare$

Proof of Theorem 3.11: Fiber of x in red, fiber of x' in green.