As examples of branched coverings we consider now Riemann surfaces of algebraic functions. Let us construct the Riemann surface of the function $\sqrt{z}$. This function is multivalued, that is for each z we have two choices for the value of $\sqrt{z}$, so we don't have a function in the usual sense. We shall construct a Riemann surface on which $\sqrt{z}$ becomes an usual holomorphic function. To see the problem, set

$$f_1, f_2 : \mathbb{C}^* \to \mathbb{C}, \quad f_1(z) = \sqrt{|z|}\, e^{i\frac{\arg z}{2}}, \quad f_2(z) = \sqrt{|z|}\, e^{i\left(\frac{\arg z + 2\pi}{2}\right)} = -f_1(z)$$

where $\arg : \mathbb{C}^* \to (-\pi, \pi]$ is the main branch of the argument. Then $\sqrt{z} \in \{f_1(z), f_2(z)\}$. The argument is continuous on $\mathbb{C} \setminus \mathbb{R}_-$ but is not continuous on the negative axis $\mathbb{R}_-$ and has a jump of 2π for $a \in \mathbb{R}_-$:

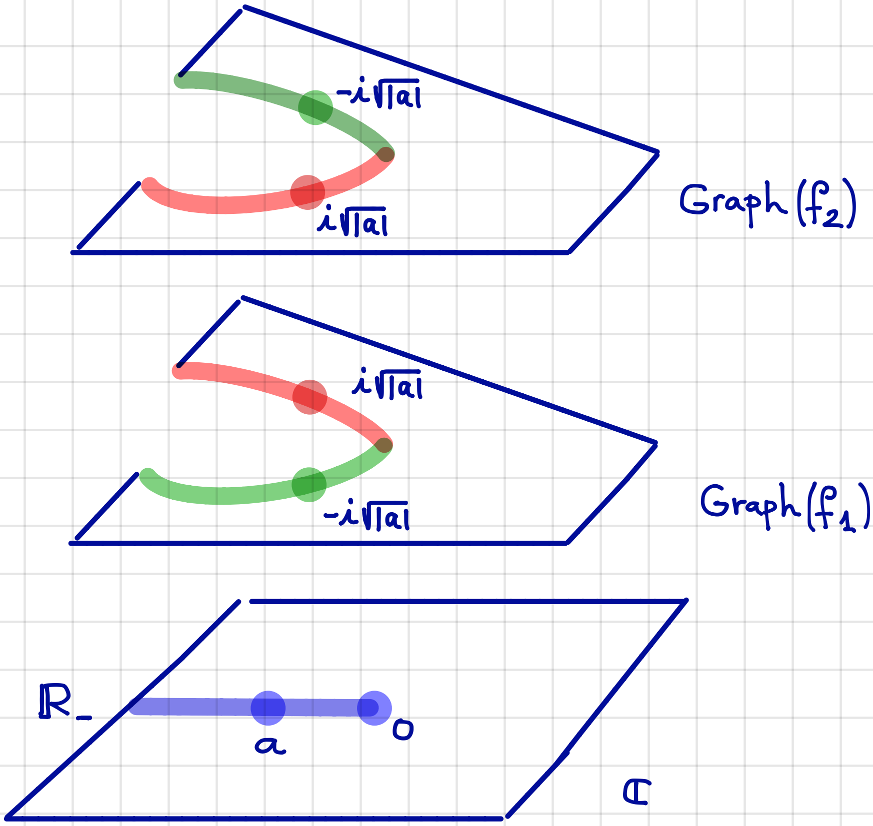
$$\lim_{\substack{\operatorname{Re} z \to a \\ \operatorname{Im} z > 0}} \arg z = \pi, \qquad \lim_{\substack{\operatorname{Re} z \to a \\ \operatorname{Im} z < 0}} \arg z = -\pi.$$

Hence

$$\lim_{\substack{\operatorname{Re} z \to a \\ \operatorname{Im} z > 0}} f_1(z) = i\sqrt{|a|} = \lim_{\substack{\operatorname{Re} z \to a \\ \operatorname{Im} z < 0}} f_2(z)$$

$$\lim_{\substack{\operatorname{Re} z \to a \\ \operatorname{Im} z < 0}} f_1(z) = -i\sqrt{|a|} = \lim_{\substack{\operatorname{Re} z \to a \\ \operatorname{Im} z > 0}} f_2(z).$$

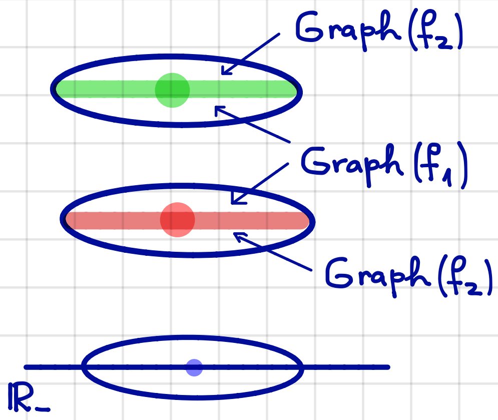
There is no continuous extension of the functions f_1 or f_2 over the half-line $\mathbb{R}_-$. The graphs of f_1 and f_2 have a "cut" over $\mathbb{R}_-$. Riemann's idea was to extend the function f_1 by the passage over $\mathbb{R}_-$ with the function f_2 by glueing the two graphs.



The two graphs are two copies of $\mathbb{C} \setminus \mathbb{R}_-$, a homeomorphism is given by $z \longmapsto (z, f_j(z)), j=1,2$. We glue now the upper side of the cut on the first copy with the lower side of the cut on the second copy (in red) and the lower side on the first copy to the upper side of the second. We can do this without self-intersections and we obtain a surface X which is the union of two copies of $\mathbb{C} \setminus \mathbb{R}_-$ glued as above. This is indeed a Riemann surface. We have the charts $(z, f_j(z)) \longmapsto z$ outside the cuts, but also in the neighbourhood of the cut points $(a, f_j(a))$ with $a \in \mathbb{R}_-$ we have parametrizations of the form $B_r(a) \longrightarrow X$

$$z \longmapsto \begin{cases} (z, f_1(z)), & \text{if } \operatorname{Im} z > 0 \\ (z, f_2(z)), & \text{if } \operatorname{Im} z < 0 \end{cases} \qquad z \longmapsto \begin{cases} (z, f_2(z)), & \text{if } \operatorname{Im} z > 0 \\ (z, f_1(z)), & \text{if } \operatorname{Im} z < 0 \end{cases}$$

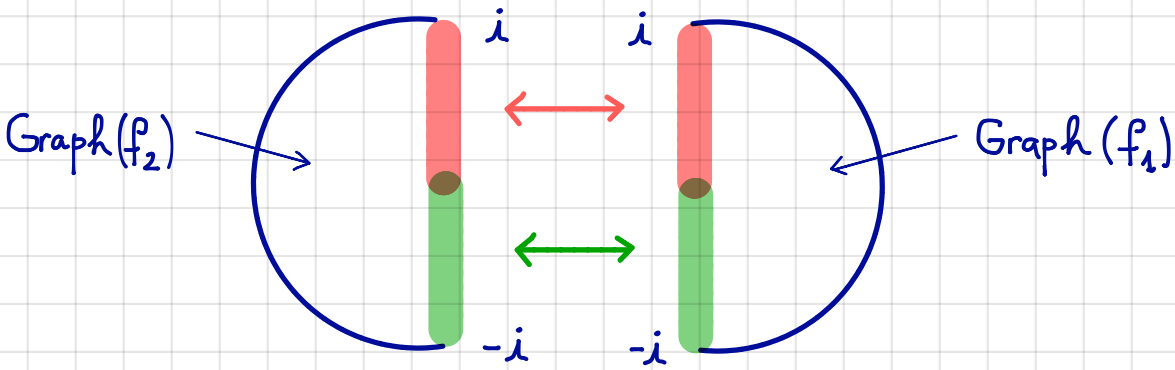
which are well-defined by the way we glued the graphs. We could actually choose any two continuous determination $f_3 : B_r(a) \to \mathbb{C}$ of the square root.

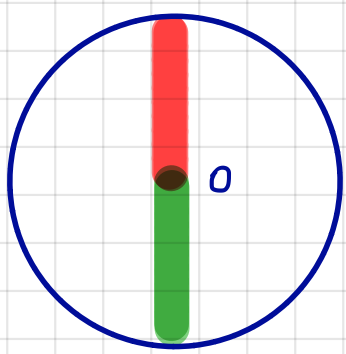


The inverse of these parametrizations give charts on $X \setminus \{0\}$, where $0 \in X$ the point obtainded by gluing the origins of the two planes, corresponding to the point $(0, \sqrt{0}) = (0,0)$ of the graphs.

In the neighbourhood of 0 we construct the following chart.

First oberve in the neighbourhood of $(0,0)$ the graphs of f_1 and f_2 are homeomorphic to half-discs by the maps $(z, f_j(z)) \longmapsto f_j(z)$.





By gluing the two sets we obtain a neighbourhood of 0 in X which is homeomorphic to a disc by the map $(z, f_1(z)) \longrightarrow f_1(z)$ on the right half-disc and $(z, f_2(z)) \longmapsto f_2(z)$ on the left disc.
Using this chart near 0 and the charts described above on $X \setminus \{0\}$ we form a holomorphic atlas on X which becomes a Riemann surface.
Let us introduce the map $p : X \to \mathbb{C}$, $p(z, f_j(z)) = z$. The charts on $X \setminus \{0\}$ are local restrictions of p so it is obvious that p is holomorphic on $X \setminus \{0\}$. The local expression of p in the chart near 0 is $f_j(z) \longmapsto (z, f_j(z)) \longmapsto z$ also $w \longmapsto w^2$, thus p is holomorphic on X. Moreover, p is a branched covering of degree two with branching point 0.
Let us introduce the map

$$f : X \to \mathbb{C}, \qquad f(z, f_j(z)) = f_j(z), \quad j = 1, 2.$$

This map is continuous near the red and green lines on X since there the function f_1 extends continuosly into f_2. Using the atlas above we see easily that f is holomorphic on X. We have $f_j^2(z) = z$ thus f is the single-valued holomorphic function on the Riemann surface X representing the multivalued function $\sqrt{z}$ on $\mathbb{C}$.

We examine next the Riemann surface of $\sqrt{z}$ from a more analytic point of view. Let us try to construct the graph of $w=\sqrt{z}$.
Now $w=\sqrt{z}$ is equivalent to $w^2=z$ so let us consider the set

$$X=\{(z,w)\in\mathbb{C}^2 : z=w^2\}=\{(w^2,w)\in\mathbb{C}^2 : w\in\mathbb{C}\}$$

This is the graph of the function $w\longmapsto w^2$ over the w-axis.
The projection $p_2: X\to\mathbb{C}$, $p_2(z,w)=w$ is well defined and due to $p_2(w^2,w)=w$, this can be considered as the function $z\mapsto\sqrt{z}$.
It is clear that X is a Riemann surface. We an atlas formed by only one chart, namely p_2. The inverse is $p_2^{-1}(w)=(w^2,w)$.
With respect to this structure the function p_2 is holomorphic and X is actually biholomorphic to $\mathbb{C}$. The projection $p_1: X\to\mathbb{C}$, $p_1(z,w)=z$ is also holomorphic, actually a branched covering with two sheets and a branching point at $O=(0,0)$.
We wish to examine the Riemann surface X at infinity, which means for $|z|, |w|$ very large. Consider the set $U=\{w\in\mathbb{C}: |w|>1\}$ and $X_U=\{(z,w): w\in U\}$. We define the chart $\varphi: X_U\longrightarrow\mathbb{C}$ $\varphi(w^2,w)=\frac{1}{w}$, which is clearly compatible with p_2. Moreover, the image of X_U is the punctered unit disc $\mathbb{D}^*=\{u\in\mathbb{C}: 0<|u|<1\}$. and $\varphi(w^2,w)\longrightarrow 0$ for $w\longrightarrow\infty$, i.e. when (w^2,w) tends to "infinity" on X. The idea occurs to introduce a point ∞_X at infinity on X which should be mapped to 0 by φ. We take a point $\infty_X\notin X$

and form the union $\widehat{X} = X \cup \{\infty_X\}$, the one point compactification of X. We introduce a topology on $\widehat{X}$ by adding to the usual open sets of X the open neighbourhoods of ∞_X which are by definition of the form $\{(z,w) \in X : |w| > R\} \cup \{\infty_X\}$ for any $R \geq 0$. The map

$$\hat{\varphi} : \widehat{X}_U = X_U \cup \{\infty_X\} \to \mathbb{D}, \quad \hat{\varphi}(a) = \begin{cases} \frac{1}{w} & \text{if } a = (w^2, w), \\ 0 & \text{if } a = \infty_X \end{cases}$$

is a homeomorphism and we use it as a chart at infinity on $\widehat{X}$. Together with p_2 they form an atlas on $\widehat{X}$, which becomes a compact Riemann surface.

The map $\hat{p}_2 : \widehat{X} \to \hat{\mathbb{C}} = \hat{\mathbb{C}}_w$, defined by extending p_2 with $\hat{p}_2(\infty_X) = \infty_w$ is biholomorphic, thus $\widehat{X}$ is biholomorphic to $\hat{\mathbb{C}}$.

We extend the function p_1 to $\hat{p}_1 : \widehat{X} \to \hat{\mathbb{C}}_z$ by setting $\hat{p}_1(\infty_X) = \infty_z$.

Using the chart $z \mapsto \frac{1}{z}$ in the neighbourhood of ∞_z, the local form of $\hat{p}_2$ is $\mathbb{D} \to \mathbb{D}$, $w \mapsto \begin{Bmatrix} \left(\frac{1}{w^2}, \frac{1}{w}\right), & w \neq 0 \\ \infty_X & , w = 0 \end{Bmatrix} \mapsto \frac{1}{w^2} \mapsto w^2$.

Thus $\hat{p}_1$ is holomorphic and is a branched covering of $\hat{\mathbb{C}}$ with two sheets and two branching points at $(0,0)$ and ∞_X.