4. Riemann surfaces as algebraic curves

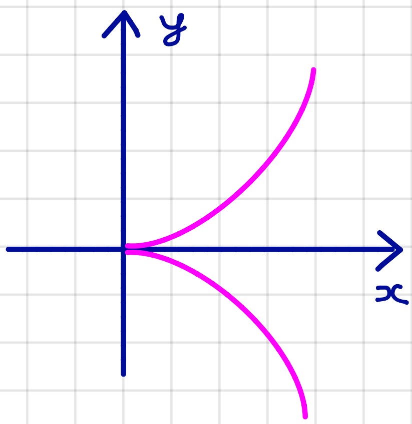
The goal of this paragraph is to study Riemann surfaces given by polynomial equations in the affine space $\mathbb{C}^2$ and the projective space $\mathbb{P}^2$.

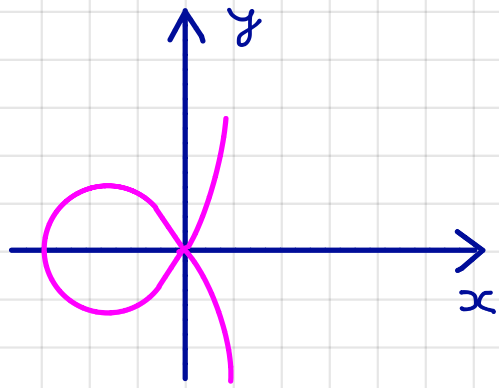
Definition 4.1 A plane affine algebraic curve is the zero set in $\mathbb{C}^2$ of a polynomial $P \in \mathbb{C}[z,w]$ in two complex variables:

$$V(P) = \{ (z,w) \in \mathbb{C}^2 : P(z,w) = 0 \}.$$

If P has no multiple factors in its decomposition in prime factors then $V(P)$ is called a curve of degree $\deg P$. The curves of degree one are the (complex) lines, the curves of degree two are called quadrics (circle, ellipse, parabola, hyperbola), the curves of degree three are called cubics, those of degree four quartics, those of degree five are cuintics and so on.

As examples, $\{(z,w) : w = z^2\}$ is called a parabola, $\{(z,w) : z^3 = w^2\}$ is called Neil's parabola, $\{(z,w) : w^2 = z^3 + z^2\}$ is called Newton's knot.





Note that a polynomial P determines completely $V(P)$ as a set, but $V(P)$ doesn't determine P. For example, $V(z^2) = V(z)$ is the w-axis. More generally, $V(P_1^{k_1} \dots P_n^{k_n}) = V(P_1 \dots P_n)$ for every polynomials $P_1, \dots, P_n$. This is why in the definition of the degree of a curve we consider only polynomials without multiple irreducible factors. It is easy to see that $V(PQ) = V(P) \cup V(Q)$.

A curve C is called <u>reducible</u> if it is the union $C = C_1 \cup C_2$ of two curves C_1 and C_2, where $C_1, C_2 \neq C$.

Otherwise C is called <u>irreducible</u>.

<u>4.2 Study's Lemma</u> Let $P \in \mathbb{C}[z,w]$ be irreducible and let $Q \in \mathbb{C}[z,w]$ such that $V(P) \subset V(Q)$, that is Q vanishes on $V(P)$. Then P divides Q in $\mathbb{C}[z,w]$.

The assertion is very easy for polynomials in one variable, since for $Q \in \mathbb{C}[z]$, $Q(a) = 0$ implies that $z - a$ divides Q. In $\mathbb{C}[z,w]$ the proof is more involved. Actually, Study's Lemma is a particular case of a central result in algebraic geometry, Hilbert's Nullstellensatz. An immediate consequence of Study's Lemma is the following:

<u>Corollary</u> A curve C is irreducible if and only if there exists an irreducible polynomial P with $C = V(P)$.

In the language of algebraic geometry this can be formulated as folows. Consider the set $\mathcal{J}(C)=\{Q\in\mathbb{C}[z,w]: C\subset V(Q)\}$ of polynomials vanishing on C. It is easy to check that $\mathcal{J}(C)$ is an ideal in $\mathbb{C}[z,w]$, called the ideal of C. Then we have:

A curve C is irreducible if and only if there exists an irreducible polynomial $P\in\mathbb{C}[z,w]$ with $\mathcal{J}(C)=(P)$

To prove the assertion, assume that $C=V(Q)$ is irreducible. Since $\mathbb{C}[z,w]$ is a unique factorization domain, $Q=q_1^{k_1}\dots q_m^{k_m}$, where $q_1,\dots,q_m$ are irreducible and coprime. We have $V(Q)=V(q_1)\cup\dots\cup V(q_m)$. If $m>1$, then $V(q_i)\neq V(Q)$ for any i. Indeed, if say $V(q_1)=V(Q)\supset V(q_2)$, then $q_2\mid q_1$ by Study's Lemma, which is impossible. We obtain a decomposition of $V(Q)$ in proper subcurves, which is a contradiction.

Conversely, if $C=V(P^k)$ with P irreducible, and $C=C_1\cup C_2$ with $C_1=V(P_1)$ and $C_2=V(P_2)$ then for any irreducible factor p_j of P_j we have $V(P)=V(P^k)\supset V(P_j)\supset V(p_j)$. By Study's Lemma $p\mid P$ so $P=cp$, for some $c\in\mathbb{C}^*$. Therefore $V(P)=V(P_j)$, for $j=1,2$.

Corollary If C is a curve then there are irreducible curves $C_1,\dots,C_m$ such that $C=C_1\cup\dots\cup C_m$ and $C_i\not\subset C_j$ for $i\neq j$. Moreover, if $D\subset C$ is an irreducible curve then $D=C_i$ for some i. We call $C_1,\dots,C_m$ the irreducible components of C.

Let $P = P_1^{k_1} \dots P_n^{k_n}$ the decomposition in irreducible factors of $P \in \mathbb{C}[z,w]$. Then $V(P) = V(P_1) \cup \dots \cup V(P_n)$ and for $i \neq j$ we have $V(P_i) \neq V(P_j)$ by Study's Lemma.

If $D \subset C$ is an irreducible curve, then $C = V(Q)$ for some irreducible polynomial Q. Since $V(Q) \subset V(P)$, Q divides P. Since $K[z,w]$ is is a unique factorization domain, $Q = cP_i$ for some i and $c \in \mathbb{C}^*$. Thus $D = C_i$.

Due to the decomposition in irreducible components, we mainly study irreducible curves.

<u>Proof of Study's Lemma</u> We use the following two claims.

<u>Claim 1</u> For any nonconstant $P \in \mathbb{C}[z,w]$ the set $V(P)$ is infinite.

Write $P(z,w) = w^n P_n(z) + w^{n-1} P_{n-1}(z) + \dots + P_0(z)$. If $P(z,w) = P_0(z)$ with $\deg P_0 \geq 1$, then $(z,w) \in V(P)$ for any root z of P and any $w \in \mathbb{C}$. If $\deg_w P \geq 1$ then for any fixed $z \in \mathbb{C}$, the polynomial $P(z,w) \in \mathbb{C}[w]$ has at least one zero (in fact $\deg_w P$). Since there are infinitely many $z \in \mathbb{C}$, this proves the claim.

<u>Claim 2</u> Let $P, Q \in \mathbb{C}[z,w]$ be coprime. Then $V(P) \cap V(Q)$ is finite.

To show this we view P, Q as elements of $\mathbb{C}(z)[w]$, where $\mathbb{C}(z)$ is the field of rational functions in z. By the Gauss Lemma, P and Q are coprime in $\mathbb{C}(z)[w]$. Since $\mathbb{C}(z)$ is a field, $\mathbb{C}(z)[z]$ is an Euclidean ring, so P, Q being coprime there exists $r = r(z) \in \mathbb{C}(z)$ and $a, b \in \mathbb{C}(z)[w]$

with

$$\begin{aligned} r(z) &= a(z,w)\,P(z,w) + b(z,w)\,Q(z,w) \\ &= \frac{a_0(z,w)}{a_1(z)}\,P(z,w) + \frac{b_0(z,w)}{b_1(z)}\,Q(z,w) \end{aligned}$$

wit polynomials a_0, a_1, b_0, b_1. After clearing denominators in z we get an equation of polynomials

$$R(z) = A(z,w)\,P(z,w) + B(z,w)\,Q(z,w).$$

Now for any $(z,w) \in V(P) \cap V(Q)$ we have $R(z) = 0$, which can happen only for finitely many values of z. Repeating this argument with the roles of z and w switched we find that there are finitely many values of w such that $(z,w) \in V(P) \cap V(Q)$. This proves the claim.

We prove now Study's Lemma. If P divides Q it is clear that $V(P) \subset V(Q)$. If conversely $V(P) \subset V(Q)$ then $V(P) \cap V(Q) = V(P)$ which is infinite by Claim 1. Claim 2 implies that P, Q are not coprime. Since P is irreducible, we conclude that P divides Q. ▨

Algebraic curves are in general not smooth so we look for criteria to find the smooth points.

<u>4.4 Definition</u> Let $U \subset \mathbb{C}$ be open and $\phi : U \to \mathbb{C}^2$ be holomorphic that is $\phi = (\phi_1, \phi_2)$ with $\phi_1, \phi_2 : U \to \mathbb{C}$ holomorphic. The map ϕ is called <u>parametrization</u> if $\phi : U \to \phi(U)$ is a homeomorphism and $\phi'(z) = (\phi_1'(z), \phi_2'(z)) \neq (0,0)$ for every $z \in U$.

Let $D \subset \mathbb{C}^2$ be open. A function $f : D \to \mathbb{C}$ is called holomorphic if it is of class $\mathcal{C}^1$ as function of four real variables and holomorphic in each complex variable, i.e.

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ and } \frac{\partial f}{\partial \bar{w}} = 0 \text{ in } D.$$

Examples are polynomials $P(z,w) \in \mathbb{C}[z,w]$ and rational functions $R(z,w) \in \mathbb{C}(z,w)$ on their domain of definition.

<u>4.3 Theorem (Implicit function Theorem)</u> Let $(z_0, w_0) \in V(P)$ such that $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$. Then there exist discs $B_1 = B_{r_1}(z_0)$, $B_2 = B_{r_2}(w_0)$ in $\mathbb{C}$ and a holomorphic function $\varphi : B_1 \to B_2$ such that

$$V(P) \cap (B_1 \times B_2) = \operatorname{Grap} \varphi = \{(z, \varphi(z)) : z \in B_1\}$$

that is $V(P)$ is the graph of a holomorphic function $\varphi = \varphi(z)$ near (z_0, w_0).

If $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$, then in some neighbourhood $B_1 \times B_2$ of (z_0, w_0), $V(P)$ is the graph of a holomorphic function $\psi = \psi(w)$:

$$V(P) \cap (B_1 \times B_2) = \operatorname{Grap} \psi = \{(\psi(w), w) : w \in B_2\}.$$

If $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$ and $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$ then the two holomorphic functions are inverse to each other.

<u>Proof</u> We apply the implicit function theorem for real maps. Denote $z = x + iy$, $w = u + iv$, so $f(z,w) = f(x,y,u,v)$ is a map from $\mathbb{R}^4$ to $\mathbb{R}^2$. The functional determinant $\left|\frac{\partial(f_1, f_2)}{\partial(u, v)}\right|$ wrt respect to u, v is the

Jacobi matrix of the map $(u,v) \longmapsto (f_1(u,v), f_2(u,v))$, which is holomorphic. By the Cauchy-Riemann equations for the variable w,

$$\left|\frac{\partial(f_1,f_2)}{\partial(u,v)}\right| = \left|\frac{\partial f}{\partial w}\right|^2$$

Applying the implicit function theorem we obtain a function φ of class C^1 such that $V(P)$ is locally its graph. Then $f(z,\varphi(z))=0$ thus

$$0 = \frac{\partial}{\partial \bar{z}} f(z,\varphi(z)) = \frac{\partial}{\partial \bar{z}} f(z,\bar{z},\varphi(z),\overline{\varphi(z)})$$

$$= \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}}\cdot\frac{\partial \bar{z}}{\partial \bar{z}} + \frac{\partial f}{\partial w}\cdot\frac{\partial \varphi}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\cdot\frac{\partial \bar{\varphi}}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial w}\cdot\frac{\partial \varphi}{\partial \bar{z}}$$

since $\frac{\partial f}{\partial \bar{z}} = 0 = \frac{\partial f}{\partial \bar{w}}$ and $\frac{\partial z}{\partial \bar{z}} = 0$. Now $\frac{\partial f}{\partial w}(z,\varphi(z)) \neq 0$ on B_1 so $\frac{\partial \varphi}{\partial \bar{z}} = 0$ on B_1, that is, φ is holomorphic. $\blacksquare$

4.5 Theorem Let $X \subset \mathbb{C}^2$. The following are equivalent:

(i) For any point $(z_0,w_0) \in X$ there exists an open set $D \subset \mathbb{C}^2$ and a parametrisation $\phi: U \to X \cap D$.

(ii) For any point $(z_0,w_0) \in X$ there exists an open set $D \subset \mathbb{C}^2$ and a holomorphic map $f: D \to \mathbb{C}$ such that $\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right) \neq (0,0)$ on D and $X \cap D = f^{-1}(c)$ for some $c \in \mathbb{C}$.

(iii) X is locally the graph of a holomorphic function over the z-axis or over the w-axis.

The proof follows from the implicit function theorem. For example if $\frac{\partial P}{\partial z}(z_0,w_0) \neq 0$ the map $\phi: B_1 \to X \cap (B_1 \times B_2)$, $z \longmapsto (z,\varphi(z))$ is a homeomorphism and $\phi'(z)=(1,\varphi'(z))\neq(0,0)$. If $\frac{\partial P}{\partial w}(z_0,w_0)\neq 0$ the same holds for $B_2\longrightarrow X\cap(B_1\times B_2)$, $w\longmapsto(\psi(w),w)$.

If one of the conditions (i)-(iii) are fulfilled we say that X is a complex submanifold of dimension one of $\mathbb{C}^2$.

4.6. Theorem Let X be a connected complex submanifold of dimension one of $\mathbb{C}^2$. Then the projections on the z-axis and w-axis induce local homeomorphism onto open sets of $\mathbb{C}$ which form a holomorphic atlas. With this structure X is a Riemann surface, called embedded Riemann surface in $\mathbb{C}^2$.

Proof Due to 4.5 (iii) for every point $(z_0,w_0)\in X$ there exist $B_1=B_{r_1}(z)$, $B_2=B_{r_2}(w)$ in $\mathbb{C}$ such that one of the two situations occur:

(a) There exists a holomorphic function $\varphi: B_1\to B_2$ with $X\cap(B_1\times B_2)=$ Graph(φ). We define a chart on X by $X\cap(B_1\times B_2)\ni(z,\varphi(z))\longmapsto z\in B_1$.

(b) There exists a holomorphic function $\psi: B_2\to B_1$ with $X\cap(B_1\times B_2)=$ Graph$(\psi)=\{(\psi(w),w): w\in B_2\}$. A chart is given by $(\psi(w),w)\longmapsto w$.

We have thus a collection of charts of two kinds which cover X. We have to check that the transition maps are holomorphic. For charts of the first kind the tranzitions have the form $z\longmapsto z$, and between second kind the form $w\longmapsto w$, on appropriate intersections of discs. The transition between a chart of first kind and second kind has the form $z \longmapsto (z, \varphi(z)) \longmapsto \varphi(z)$ and this is holomorphic. ▨

We deduce immediately the following.

<u>Theorem 4.7</u> Let $P \in \mathbb{C}[z,w]$, $D \subset \mathbb{C}^2$ open and $X = V(P) \cap D$. Assume that at any point $(z,w) \in X$ we have $\left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}\right) \neq (0,0)$. Then each connected component of X has a structure of a Riemann surface.

<u>Definition 4.8</u> Let C be an algebraic curve in $\mathbb{C}^2$.
A point $(z_0, w_0) \in C$ is called <u>regular</u> if there exists an open neighbourhood $D \subset \mathbb{C}^2$ of (z_0, w_0) such that $C \cap D$ is a complex submanifold of $\mathbb{C}^2$ of dimension one. A point $(z_0, w_0) \in C$ is called <u>singular</u> if it is not regular. The set of regular/singular points is denoted by $\operatorname{Reg} V(P) / \operatorname{Sing} V(P)$.

If $(z_0, w_0) \in C$ regular there exists an open set $D \subset \mathbb{C}^2$ and a parametrisation $\phi : U \to C \cap D$.
Every connected component of $\operatorname{Reg} V(P)$ is a Riemann surface. The charts are given by the inverses of the parametrizations $\phi^{-1} : \phi(U) \to U$.
To check the compatibility of these charts requires some extra work.
We know that $\{(z,w) \in V(P) : \left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}\right) \neq (0,0)\} \subset \operatorname{Reg} V(P)$ and $\operatorname{Sing} V(P) \subset \{(z,w) : \left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}\right) = (0,0)\}$ but in general the inclusions

are strict.

<u>Examples</u> For $P(z,w) = w - z^2$ all points of $V(P)$ are regular.

For $P(z,w) = zw$, $V(P)$ is the union of the z-axis and w-axis.

$\operatorname{Sing} V(P) = \{(0,0)\}$ und $\operatorname{Reg} V(P) = V(P) \setminus \{(0,0)\}$.

For $P(z,w) = z^2$, $V(P)$ is the w-axis thus $\operatorname{Sing} V(P) = \emptyset$ although $\frac{\partial P}{\partial z}(0,0) = \frac{\partial P}{\partial w}(0,0) = 0$.

<u>4.4. Theorem</u> Assume that $P \in \mathbb{C}[z,w]$ is irreducible. Then $\operatorname{Sing} V(P) = \{(z,w) \in V(P) : (\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}) = (0,0)\}$ and $\operatorname{Sing} V(P)$ is a finite set. Moreover, $\operatorname{Reg} V(P)$ is connected, hence a Riemann surface.

In particular, if P is irreducible and the set $\{P = \frac{\partial P}{\partial z} = \frac{\partial P}{\partial w} = 0\}$ is empty, then $V(P)$ is an embedded Riemann surface in $\mathbb{C}^2$.