We wish now to consider the behaviour of algebraic curves at infinity. The algebraic trick which leads to the solution is to introduce <u>homogeneous coordinates</u> z_0, z_1, z_2 with $z = \frac{z_1}{z_0}$, $w = \frac{z_2}{z_0}$ and to make $z_0 = 0$, which sends z, w to infinity on the complex line of equation $w = \frac{z_2}{z_1} z$ in $\mathbb{C}^2$. We add in this way a point at infinity to each complex line through the origin in $\mathbb{C}^2$. The union of these infinity points forms a projective line at infinity L_∞, given by the points of homogeneous coordinates $0, z_1, z_2$. Note that if z_0, z_1, z_2 are the homogeneous coordinates of a point z, w then also $\lambda z_0, \lambda z_1, \lambda z_2$ are if $\lambda \in \mathbb{C}^*$. Thus points of homogeneous coordinates $0, z_1, z_2$ indeed form a projective line $\mathbb{P}^1$.

Let us determine in this way the points at infinity of the curve $w = z^2$. In homogeneous coordinates this becomes $z_1^2 = z_0 z_2$ and setting $z_0 = 0$ we obtain $z_1 = 0$. There is only one point at infinity of homogeneous coordinates $0, 0, z_2$ or equivalently $0, 0, 1$.

Let us define rigorously the space $\mathbb{C}^2 \cup L_\infty = \mathbb{C}^2 \cup \mathbb{P}^1$. Since z_0, z_1, z_2 represent the same point as $\lambda z_0, \lambda z_1, \lambda z_2$, $\lambda \in \mathbb{C}^*$ we proceed as in the construction of the projective space $\mathbb{P}^1$. Consider on $\mathbb{C}^3 \setminus \{0\}$ the equivalence relation $Z_1 \sim Z_2$ if and only if there exists $\lambda \in \mathbb{C}^*$ with $Z_2 = \lambda Z_1$. The quotient space $\mathbb{C}^3 \setminus \{0\} / \sim$ is denoted by $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ and is called the two dimensional complex projective space. Let $\pi: \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$, $\pi(z_0, z_1, z_2) := [z_0, z_1, z_2]$ be the canonical projection. We call z_0, z_1, z_2 the homogeneous coordinates of the the points on the line $\pi(z_0, z_1, z_2)$.

We introduce on $\mathbb{P}^2$ the quotient topology with which $\mathbb{P}^2$ becomes a compact connected space. Moreover, $\mathbb{P}^2$ is locally homeomorphic to $\mathbb{C}^2$. Consider the open sets $U_j = \{[z] \in \mathbb{P}^2 : z_j \neq 0\}$, $j = 0, 1, 2$. We define the chart $\varphi_0 : U_0 \to \mathbb{C}^2$, $\varphi_0([z]) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$ which is a homeomorphism with inverse $\varphi_0^{-1}(z, w) = [1, z, w]$. We define the the charts $\varphi_j : U_j \to \mathbb{C}^2$ for $j = 1, 2$ analogously by dividing with z_j the other two homogeneous coordinates.

It is easy to check that the transition maps $\varphi_j \circ \varphi_k^{-1}$ have components rational functions hence are biholomorphic maps of two complex variables. Thus $\mathbb{P}^2$ is a complex manifold of dimension two. The complement of U_0 in $\mathbb{P}^2$ is $L_\infty = \{[z] \in \mathbb{P}^2 : z_0 = 0\}$ which is homeomorphic to $\mathbb{P}^1$ by $[0, z_1, z_2] \mapsto [z_1, z_2]$. The same holds for $\mathbb{P}^2 \setminus U_j$ for $j = 1, 2$, too.

We return now to algebraic curves. In order to study curves in the projective space we need homogeneous polynomials.

A polynomial $Q \in \mathbb{C}[u_1, \dots, u_r]$ is called homogeneous of degree n if $Q(\lambda u) = \lambda^n Q(u)$ for any $\lambda \in \mathbb{C}^*$, that is, if all monomials of Q have degree n:

$$Q(u_1,\dots,u_r)=\sum_{\alpha_1+\dots+\alpha_r=n} c_{\alpha_1\dots\alpha_r}\, u_1^{\alpha_1}\dots u_r^{\alpha_r}$$

Let $p\in\mathbb{C}[z,w]$ written as $p(z,w)=p_n(z,w)+p_{n-1}(z,w)+\dots+p_0$, where p_k is homogeneous of degree k (the sum of monomials $c_{lm}z^l w^m$ with $l+m=k$).

By replacing $z=z_1/z_0$, $w=z_2/z_0$ and multiplying by z_0^n in order to eliminate the denominators the equation $p(z,w)=0$ becomes

$$P(z_0,z_1,z_2):=z_0^n\, p\left(\frac{z_1}{z_0},\frac{z_2}{z_0}\right)$$

$$=z_0^n p_n\left(\frac{z_1}{z_0},\frac{z_2}{z_0}\right)+z_0^n p_{n-1}\left(\frac{z_1}{z_0},\frac{z_2}{z_0}\right)+\dots+z_0^n p_0=0$$

This polynomial has the nice feauture of being homogeneous!

Indeed, if $p_k(z,w)=\sum_{l+m=k} c_{lm}z^l w^m$, then

$$z_0^n p_k\left(\frac{z_1}{z_0},\frac{z_2}{z_0}\right)=\sum_{l+m=k} c_{lm}\, z_0^{n-k} z_1^l z_2^m$$

is homogeneous of degree n. The polynomial P above is called the <u>homogenization</u> of p.

Given a homogeneous polynomial $Q\in\mathbb{C}[z_0,z_1,z_2]$, the polynomial $q(z,w)=Q(1,z,w)$ is called the <u>dehomogenization</u> of Q.

We have thus a correspondence between homogeneous and inhomogeneous polynomials:

$$p(z,w)\longmapsto z_0^{\deg p}\, p\left(\frac{z_1}{z_0},\frac{z_2}{z_0}\right)=P(z_0,z_1,z_2)$$

$$q(z,w)=Q(1,z,w)\longleftarrow Q(z_0,z_1,z_2)$$

Note that $\deg q = \deg Q$ if and only if z_0 does not divide Q. In this case the homogenization of q is Q. It is clear that $P(1,z,w) = p(z,w)$. Thus the above correspondence is bijective between the following:

$$\mathbb{C}[z,w] \longleftrightarrow \{P \in \mathbb{C}[z_0,z_1,z_2] : P \text{ homogeneous}, z_0 \nmid P\}$$

4.9 Definition A plane projective algebraic curve is a set

$$V(Q) = \{[z_0,z_1,z_2] \in \mathbb{P}^2 : Q(z_0,z_1,z_2) = 0\}$$

where $Q \in \mathbb{C}[z_0,z_1,z_2]$ is a nonconstant homogeneous polynomial. If Q has no multiple irreducible factors then we call $\deg Q$ the degree of the curve $V(Q)$. Note that Q is not well defined on $\mathbb{P}^2$ but the condition $Q(z_0,z_1,z_2)=0$ is independent on the choice of homogeneous coordinates since $Q(\lambda z_0, \lambda z_1, \lambda z_2) = 0$ if and only if $Q(z_0,z_1,z_2)=0$ due to the homogeneity of Q.

4.10 Theorem A projective algebraic curve is compact.

Proof The set $V(Q)$ is closed in $\mathbb{P}^2$ since its preimage by the projection $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ is the closed set $\{z \in \mathbb{C}^3 \setminus \{0\} : Q(z) = 0\}$. Being closed in the compact space $\mathbb{P}^2$ the set $V(Q)$ is compact. $\blacksquare$

We have the following one to one correspondence:

$$\{\text{Affine algebraic curves}\} \longleftrightarrow \left\{\begin{array}{l}\text{Projective algebraic curves} \\ \text{not containing the line} \\ \text{at infinity } L_\infty\end{array}\right\}$$

$$V(q) \longleftrightarrow V(Q)$$

Note that for $Q = z_0^n$ we have $q = 1$, so we have to exclude polynomials Q divisible by z_0, thus curves containing $L_\infty = \{z_0 = 0\}$.

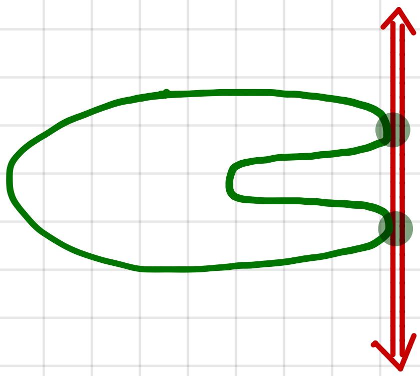
The affine part of $V(Q)$ is $V(Q) \cap U_0 \cong \varphi_0(V(Q) \cap U_0) = V(q)$:

$$V(Q) \cap U_0 = \{[z] \in U_0 : Q(z) = 0\} = \{[z] \in U_0 : z_0^{\deg Q} Q(1, \tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}) = 0\}$$
$$= \{[z] \in U_0 : Q(1, \tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}) = 0\}$$

$$\varphi_0(V(Q) \cap U_0) = \{(\tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}) : [z_0, z_1, z_2] \in V(Q) \cap U_0\} = \{(u,v) \in \mathbb{C}^2 : Q(1,u,v) = 0\}$$
$$= \{(u,v) \in \mathbb{C}^2 : q(u,v) = 0\}$$

We write shortly $V(Q) \cap \mathbb{C}^2 = V(q)$ by identifying U_0 to $\mathbb{C}^2$.

Converely, $\overline{\varphi_0^{-1}(V(q))} = V(Q)$, which we write $\overline{V(q)} = V(Q)$, that is $V(Q)$ is the compactification of $V(q)$ and $V(Q) \setminus V(q) = V(Q) \cap L_\infty$ is finite.



$\mathbb{C}^2$ $\qquad$ $L_\infty = \{z_0 = 0\} \cong \mathbb{P}^1$

If $Q = \sum_{j+k+l=n} c_{jkl} z_0^j z_1^k z_2^l$, where $n = \deg Q$, then

$$V(Q) \cap L_\infty = \{[0, z_1, z_2] : Q(0, z_1, z_2) = \sum_{k+l=n} c_{0kl} z_1^k z_2^l = 0\}$$

and $Q(0, z_1, z_2)$ is not identically zero, otherwise z_0 would divide Q.

Thus $Q_0(z_1,z_2) = Q(0,z_1,z_2)$ is a non-trivial homogeneous polynomial so $V(Q) \cap L_\infty$ can be identified by the bijective map $L_\infty \to \mathbb{P}^1$, $[0,z_1,z_2] \longmapsto [z_1,z_2]$ to $V(Q_0) = \{[z_1,z_2] \in \mathbb{P}^1 : Q_0(z_1,z_2) = 0\}$.

The affine part of $V(Q_0)$, that is $V(Q_0) \cap \{[z_1,z_2] \in \mathbb{P}^1 : z_1 \neq 0\}$, can be identified to the zero set of the polynomial $Q_0(1,z_2) \in \mathbb{C}[z_2]$ in $\mathbb{C}$, which is finite. The set $V(Q_0)$ can also contain the point at infinity $[0,1] \in \mathbb{P}^1$, so it is certainly finite.

Now $V(Q)$ has no isolated points, hence $\overline{V(\mathcal{Q})} = V(Q)$.