8. ABELIAN DIFFERENTIALS

8.1. **Differentials and homology group.** We will consider a compact Riemann surface X. Since we are in two dimensions, we have the following differential forms on X: scalar functions $F(z, \bar{z})$, 1-forms (*differentials*), which locally look like

$$\omega = f(z, \bar{z})dz + g(z, \bar{z})d\bar{z},$$

and 2-forms $v(z, \bar{z})dz \wedge d\bar{z}$. These objects are invariant under changes of coordinates from local chart to chart, so they are defined on all X. We can also decompose ω into $(1, 0)$ and $(0, 1)$ parts as $f(z, \bar{z})dz$ (and $g(z, \bar{z})d\bar{z}$) and this decomposition is invariant on X since the transition functions are holomorphic, i.e., $f(z_\alpha, \bar{z}_\alpha) = f(z_\beta, \bar{z}_\beta)\frac{dz_\alpha}{dz_\beta}$.

As a topological space the Riemann surfaces are classified by the genus g, i.e. a number of "handles".

Claim: The homology group $H_1(X, \mathbb{Z})$ of the Riemann surface is generated by $2g$ cycles $a_1, ..., a_g, b_1, ..., b_g$.

Definition **8.1.** *This basis is called canonical if the intersection numbers are*

$$a_j \circ b_l = \delta_{jl}, \quad a_j \circ a_l = b_j \circ b_l = 0. \tag{1}$$

The intersection number of two 1-cycles is ± 1, depending on the orientation of the intersection and a self-intersection number of each 1-cycle is assumed to be zero. (For a precise definition of intersection numbers see [Farkas-Kra, §III.1]).

Remark: Canonical bases are not unique. Indeed, let us represent the basis by the $2g$-dimensional vector as follows

$$\begin{pmatrix} a \\ b \end{pmatrix}.$$

Then any other canonical basis is related by the integer matrix $A \in GL(2g, \mathbb{Z})$ transformation

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix},$$

with the condition of preserving the intersection numbers Eq. , i.e.

$$J = AJA^T, \quad J = \begin{pmatrix} a \\ b \end{pmatrix} \circ \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2}$$

Hence the new basis is canonical if and only if $A \in Sp(g, \mathbb{Z})$ is a symplectic matrix.

Consider now closed differentials ω, $d\omega = 0$. Given a (canonical) basis of 1-cycles, periods of ω are well-defined

$$\int_{a_j} \omega, \quad \int_{b_j} \omega,$$

i.e. independent of the choice of paths representing the cycles, for the homological chocies of paths. This is because for any two homological closed paths a and a', we have $\int_a \omega = \int_{a'} \omega$ for any closed differential ω.

8.2. **Canonical dissection.** We will often work with the *canonical dissection* of the Riemann surface. The idea is to fix a base point P_0 and then contract the canonical basis a, b so that the cycles start and end at P_0, as illustrated on the picture below.

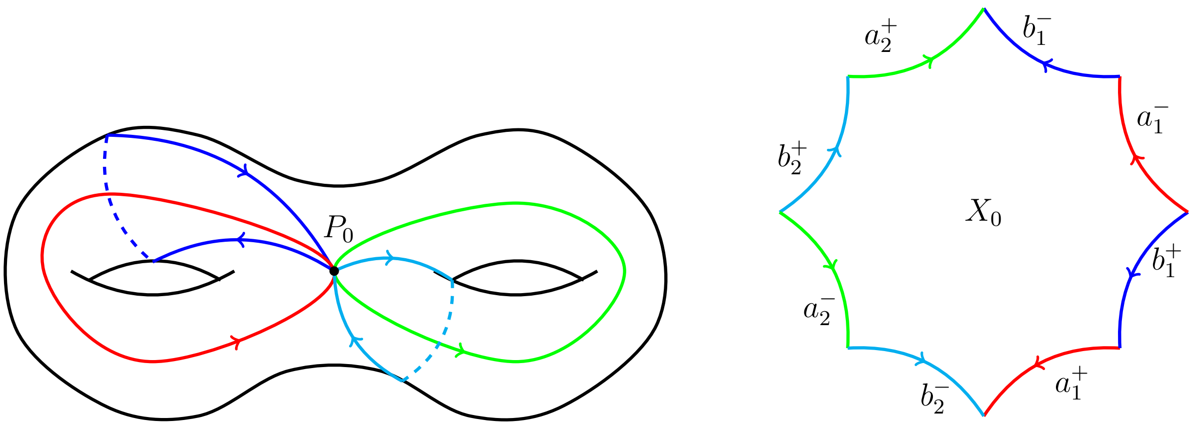

Figure 1. Riemann surface of genus 2 and its canonical dissection.

As a result we end up with the simply-connected 2-cell X_0 with the boundary

$$\partial X_0 = \sum_{j=1}^{g} \left(-a_j^+ - b_j^+ + a_j^- + b_j^-\right),$$

where a_j^+, b_j^+ (resp. a_j^-, b_j^-) are right (resp. left) sides of cuts along cycles a_j, b_j.

8.3. **Riemann's bilinear identity.**

Theorem **8.1.** *Let X be a genus-g compact Riemann surface, with the canonical basis and corresponding canonical dissection.*

(a1) For any two closed differentials ω_1, ω_2 we have

$$\int_X \omega_1 \wedge \omega_2 = \sum_{j=1}^{g} \int_{a_j} \omega_1 \cdot \int_{b_j} \omega_2 - \int_{b_j} \omega_1 \cdot \int_{a_j} \omega_2 \tag{3}$$

(a2) and

$$\int_{\partial X_0}\left(\omega_2(P)\int_{P_0}^{P}\omega_1\right)=\sum_{j=1}^{g}\int_{a_j}\omega_1\cdot\int_{b_j}\omega_2-\int_{b_j}\omega_1\cdot\int_{a_j}\omega_2, \tag{4}$$

where we assume that the contour from P_0 to P on the left side lies completely inside ∂X_0.
(b) For all holomorphic differentials ω,η

$$\sum_{j=1}^{g}\int_{a_j}\omega\cdot\int_{b_j}\eta-\int_{b_j}\omega\cdot\int_{a_j}\eta=0 \tag{5}$$

(c) and for holomorphic differential $\omega\not\equiv 0$ we have

$$\operatorname{Im}\sum_{j=1}^{g}\int_{a_j}\bar{\omega}\int_{b_j}\omega>0. \tag{6}$$

Proof. To prove (a1), we note that since X_0 is simply-connected, there exists a function f on X_0, s.t. $\omega_1=df$. Then by Stokes theorem,

$$\begin{aligned}\int_X\omega_1\wedge\omega_2=\int_{X_0}\omega_1\wedge\omega_2=\int_{X_0}df\wedge\omega_2=\int_{X_0}d(f\omega_2)=\int_{\partial X_0}f\omega_2\\ =\sum_{j=1}^{g}\left(-\int_{a_j^+}-\int_{b_j^+}+\int_{a_j^-}+\int_{b_j^-}\right)f\omega_2\\ =\sum_{j=1}^{g}\int_{a_j}(f\text{ on }a_j^- - f\text{ on }a_j^+)\omega_2+\int_{b_j}(f\text{ on }b_j^- - f\text{ on }b_j^+)\omega_2.\end{aligned}$$

Next, we note that df has no discontinuity on a_j or b_j, so f on a_j^+ and a_j^- must differ by a constant, and same for b_j^+,b_j^-. Since the path b_j connects a_j^- and a_j^+ (as can be seen from the Fig. 1), we can write the last expression as

$$\sum_{j=1}^{g}\int_{a_j}\omega_1\cdot\int_{b_j}\omega_2-\int_{b_j}\omega_1\cdot\int_{a_j}\omega_2,$$

establishing (a1).

The proof of (a2) goes along similar lines. We have

$$\int_{\partial X_0}\left(\omega_2(P)\int_{P_0}^{P}\omega_1\right)=\sum_{j=1}^{g}\left(-\int_{a_j^+}-\int_{b_j^+}+\int_{a_j^-}+\int_{b_j^-}\right)\omega_2(P)\int_{P_0}^{P}\omega_1$$

$$=\sum_{j=1}^{g}\int_{a_j}\left(\omega_2(P_{a_j^-})\int_{P_0}^{P_{a_j^-}}\omega_1-\omega_2(P_{a_j^+})\int_{P_0}^{P_{a_j^+}}\omega_1\right)$$

$$+\sum_{j=1}^{g}\int_{b_j}\left(\omega_2(P_{b_j^-})\int_{P_0}^{P_{b_j^-}}\omega_1-\omega_2(P_{b_j^+})\int_{P_0}^{P_{b_j^+}}\omega_1\right)\quad(7)$$

Let $P_{a_j^-}$ and $P_{a_j^+}$ be the points, resp., on a_j^- and a_j^+ which coincide on X, and $P_{b_j^-}$ and $P_{b_j^+}$ are the points, resp., on b_j^- and b_j^+ which coincide on X.

First we note that

$$\omega_2(P_{a_j^-})=\omega_2(P_{a_j^+}),\quad \omega_2(P_{b_j^-})=\omega_2(P_{b_j^+}).$$

Next, consulting Fig. 1. one can see that

$$\int_{P_0}^{P_{a_j^-}}\omega_1-\int_{P_0}^{P_{a_j^+}}\omega_1=\int_{P_{a_j^+}}^{P_{a_j^-}}\omega_1=-\int_{b_j}\omega_1$$

and

$$\int_{P_0}^{P_{b_j^-}}\omega_1-\int_{P_0}^{P_{b_j^+}}\omega_1=\int_{P_{b_j^+}}^{P_{b_j^-}}\omega_1=\int_{a_j}\omega_1$$

Plugging this back to (7) we obtain (a2),

$$\int_{\partial X_0}\left(\omega_2(P)\int_{P_0}^{P}\omega_1\right)=\sum_{j=1}^{g}\int_{a_j}\omega_1\cdot\int_{b_j}\omega_2-\int_{b_j}\omega_1\cdot\int_{a_j}\omega_2.$$

Now, if ω_1,ω_2 are holomorphic, then $\omega_1\wedge\omega_2=0$ and (b) follows.

Next, for holomorphic ω, there exists a holomorphic function f on X_0, s.t. $\omega=df$. We apply (3) for $\omega_1=\omega$ and $\omega_2=\bar\omega$,

$$\operatorname{Im}\sum_{j=1}^{g}\int_{a_j}\bar\omega\int_{b_j}\omega=-\frac{1}{2i}\int_X\bar\omega\wedge\omega=-\frac{1}{2i}\int_{X_0}d\bar f\wedge df$$

$$=\frac{1}{2i}\int_{X_0}|\partial f|^2dz\wedge d\bar z=\int_{X_0}|\partial f|^2dx\wedge dy>0$$

where we used some local complex coordinates $z=x+iy$ and the fact that $dx\wedge dy$ is a everywhere positive 2-form on X_0. □

Corollary **8.1.** *From Eq.* (6) *it follows that if all a -periods of a holomorphic differential ω vanish, then $\omega\equiv 0$.*

8.4. **Holomorphic differentials and period matrix.** We know from Riemann-Roch theorem (Lecture 4) that the dimension of the vector space $H^{1,0}(X)$ of holomorphic differentials on X is equal to genus $g = \dim H^{1,0}(X)$.

Example: For the sphere $g = 0$ and there are no holomorphic differentials. Consider $\hat{\mathbb{C}}$, with the charts $(\mathbb{C}\backslash\infty, z)$ and $(\mathbb{C}\backslash 0, 1/z)$. Naively the differential $z^N dz$ is holomorphic in the first chart for $N \geqslant 0$. However it becomes singular in the second chart $(1/z)^N d(1/z) = -z^{-N-2}dz$.

Example: Consider the torus $g = 1$, $T = \mathbb{C} \mod \Lambda$, $\Lambda = n\tau + m$ where $n, m \in \mathbb{Z}$. We can pull back from $\mathbb{C}$ the differential $\omega = dz$, $z \in \mathbb{C}$. Indeed $\omega(z+1) = \omega(z+\tau) = \omega$. We could try to pull back $\tilde{\omega} = f(z)dz$, where $f(z)$ is a doubly periodic and holomorphic, but such function is a constant (see Liouville theorem).

Example: See the exercise sheet 4 for the holomorphic differentials on a hyperelliptic surface.

From Cor. 8.1 it follows that for any basis ω_j of $H^{1,0}(X)$ the matrix of a-periods

$$A_{jl} = \int_{a_j} \omega_l$$

is non-degenerate and invertible. Thus we can normalize the basis of holomorphic differentials as follows

Definition **8.2.** *Given the canonical basis a_j, b_j of 1-cycles, the basis of holomorphic differentials normalized as*

$$\int_{a_j} \omega_l = \delta_{jl} \tag{8}$$

is called canonical. The matrix of b-periods of the canonical basis

$$\tau_{jl} = \int_{b_j} \omega_l$$

is called the period matrix of X.

Corollary **8.2.** *The period matrix is symmetric $\tau_{jl} = \tau_{lj}$ and the matrix $\operatorname{Im}\tau > 0$ is positive-definite.*

Proof. Symmetry immediately follows by applying (5) to $\omega = \omega_j$ and $\eta = \omega_l$.

Let α_j be a real vector and apply (6) to $\omega = \sum_j \alpha_j \omega_j$. It follows that $\operatorname{Im} \sum_{j,l} \alpha_j \tau_{jl} \alpha_l = \sum_{j,l} \alpha_j (\operatorname{Im} \tau_{jl}) \alpha_l > 0$. $\square$

8.5. **Abel map.** This is the key application of holomorphic differentials on compact Riemann surfaces.

The period matrix of X generates a lattice Λ in $\mathbb{C}^g$

$$\Lambda = \{n_j + \tau_{jl} m_l,\ n, m \in \mathbb{Z}^g\},$$

generated by the a and b - periods of holomorphic differentials.

Definition **8.3** *The Jacobean variety (equiv., Jacobian) of X is the complex torus*

$$Jac(X) = \mathbb{C}^g/\Lambda.$$

If P_0 is a base point then using holomorphic differentials we obtain the holomorphic map $X \to Jac(X)$ as follows

Definition **8.4** *The Abel map is defined as*

$$\begin{aligned} I: \quad & X \to Jac(X), \\ & P \to \left(\int_{P_0}^{P} \omega_1, ..., \int_{P_0}^{P} \omega_g \right). \end{aligned}$$

This is well-defined because the right hand side is defined modulo period integrals, i.e., modulo Λ, so its a point in $Jac(X)$.

8.6. **Abelian differentials and their properties.**

Definition **8.3.** *A differential η is called meromorphic, or equivalently, Abelian differential, if in a local coordinate z it has the form $h(z)dz$ where $h(z)$ is meromorphic function.*

Zeros and poles of local function $h(z)$ define zeroes and poles of the meromorphic differentials and the notion of order of zero or pole is well-defined, i.e. independent of the choice of local coordinates.

The residue $\mathrm{Res}_{z_0}\eta$ of the Abelian differential at a singular point z_0 is defined as the h_{-1} coefficient in the Laurent expansion around z_0,

$$h(z) = \sum_{n=n_0}^{\infty} h_n (z - z_0)^n.$$

The residue is independent of the choice of local coordinates, since it can be written in the manifestly invariant form

$$\mathrm{Res}_{z_0}\eta = \frac{1}{2\pi i} \int_{\partial B_{z_0}} \eta(z),$$

where B_{z_0} is a disk containing z_0 in the interior, s.t. η is holomorphic on $\bar{B}_{z_0} \backslash \{z_0\}$, e.g. no other singular points in the closure. The following property holds.

LEMMA **8.5.** *Let $z_1, ..., z_m$ be the singular points of the Abelian differential η, then*

$$\sum_{j=1}^{m} \mathrm{Res}_{z_j}\eta = 0.$$

Proof. Let B_j be small disk around z_j containing no other singularities in its closure. Then,

$$\sum_{j=1}^{m} \mathrm{Res}_{z_j}\eta = \frac{1}{2\pi i}\sum_{j}\int_{\partial B_j}\eta = -\frac{1}{2\pi i}\int_{X-\cup B_j} d\eta = 0, \tag{9}$$

since η is holomorphic on $X - \cup B_j$ and thus closed there. □

8.7. **Differentials of 2nd and 3rd kind.** The following terminology is commonly used:

(a) Holomorphic differentials are called Abelian differentials of the *first kind*,

(b) Meromorphic differentials with poles with vanishing residues are called Abelian differentials of the *second kind*,

(c) Meromorphic differentials with non-zero residues are called Abelian differentials of the *third kind*.

Any meromorphic differential is a combination of differentials of three types.

We have already constructed the canonical basis of differentials of the first kind. Normalized Abelian differentials of the second kind are constructed as follows. The differential of 2nd kind $\eta_P^{(N)}, N \in \mathbb{N}$ has only one singularity of order $N+1$ at $P \in X$, i.e. for a local coordinate $z, z(P) = 0$,

$$\eta_P^{(N)} = \left(\frac{1}{z^{N+1}} + O(1)\right) dz.$$

Remark: This construction depends on the choice of the local coordinate, but the order of the pole is independent of the choice of local coordinate.

Example: Consider $\eta_0^{(N)} = \frac{dz}{z^{N+1}}$ on the sphere.

The basic differential of 3rd kind η_{PQ} has only two singularities at P and Q with opposite residues

$$\mathrm{Res}_P\, \eta_{PQ} = -\mathrm{Res}_Q\, \eta_{PQ} = 1.$$

Example: Consider $\eta_{z_0 z_1} = d \log \frac{z-z_0}{z-z_1}$ on the sphere.

Note that adding holomorphic differentials to $\eta_P^{(N)}$ and η_{PQ} preserves the form of singularities. Taking into account (8), this ambiguity can be used in a straightforward way to normalize the differentials above as follows

$$\int_{a_j} \eta_P^{(N)} = 0, \quad \int_{a_j} \eta_{PQ} = 0,$$

for the a-cycles. Such differentials are called *normalized* Abelian differentials of, resp., 2nd and 3rd kind. We now have to demonstrate their existence and uniqueness.

8.8. **Divisors of meromorphic functions and Abelian differentials.**

DEFINITION **8.6** *Divisors on compact Riemann surfaces are given by formal finite sums of points*

$$D = \sum_{k=1}^{N} n_k P_k, \quad n_k \in \mathbb{Z},\ P_k \in X,$$

and the sum

$$\deg D = \sum_{k=1}^{N} = n_k$$

is called the degree of D.

Set of divisors $Div(X)$ form an Abelian group with well-defined group operation (summation) and inverse element ($D \to -D$).

DEFINITION **8.7** *Divisor is called positive, if all $n_k \geqslant 0$.*

Divisor (f) of meromorphic function f is a sum of its zeroes $P_1, .., P_N$ and poles $Q_1, ..., Q_M$ with their orders as multiplicities

$$(f) = n_1 P_1 + ... + n_N P_N - n_1' Q_1 - ... - n_M' Q_M.$$

Recall that zeroes and poles of an Abelian differential is well defined

DEFINITION **8.8** *The divisor of Abelian differential ω is the sum of its zeroes and poles*

$$(\omega) = \sum_{P \in X} \mathrm{ord}(P) P.$$

Here $\mathrm{ord}(P) = n_0$ is the order of zero or singularity in local Laurent expansion $h(z) = \sum_{n=n_0}^{\infty} h_n (z - z_P)^n$ in local coordinate at the point P and locally $\omega = h(z)dz$.

DEFINITION **8.9** *A divisor is called principal if it is a divisor of a meromorphic function.*

LEMMA **8.10.** *Divisor of a meromorphic function has degree zero* $\deg(f) = 0$.

Proof. Given a meromorphic function f, consider the Abelian differential df/f. Its residues are equal to the multiplicities of the zeroes and poles of f. By Lemma 8.5, the sum of residues is zero. □

We now note that the Abel map can be naturally extended to divisors $D = \sum_{P \in X} n_P P,\ n_P \in \mathbb{Z}$, as

$$I(D) = \sum_{P \in X} n_P \int_{P_0}^{P} \omega_j.$$

Note that if $\deg D = 0$, i.e. $D = P_1 + ... + P_N - Q_1 - ... - Q_N$, then the Abel map

$$I(D) = \sum_{m=1}^{N} \int_{Q_m}^{P_m} \omega_j$$

is independent of the base point P_0.

9. Abel theorem

9.1. **Harmonic differentials.** We have established that the divisor of meromorphic function has degree zero. In other direction, given a degree zero divisor we may ask whether it is a divisor of a meromorphic function. The goal of this section is to prove the Abel theorem, saying that the divisor is principal if and only if its degree zero and $I(D) = 0$.

This relies on a two technical results on the existence of abelian differentials and on the decomposition theorem for differentials, which we cover first.

First, we would like to prove the existence of 2nd and 3rd kind differentials. Here we sketch the argument, the full construction is based on the decomposition theorem of differentials on Riemann surface, is covered e.g. in [Jost,FarkasKra].

Let $\omega = \omega_{(1,0)} + \omega_{(0,1)}$ be the unique decomposition of the differential ω in $(1,0)$ and $(0,1)$ components. The Hodge $*$-operator is defined on the differential forms as follows

$$*\omega = -i\omega_{(1,0)} + i\omega_{(0,1)}.$$

Clearly $*^2 = -1$ and type $(1,0)$ (resp. $(0,1)$) differential form eigenspaces of $*$ with eigenvalues $-i$ (resp. i).

Let X be a Riemann surface and consider the natural scalar product on the smooth differentials

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge *\bar{\omega}_2. \tag{10}$$

The Hilbert space $L^2_1(X)$ of is the completion of the space of smooth differentials $\mathcal{E}^1(X)$ under this scalar product.

We have: $(\omega_1, \omega_2) = \overline{(\omega_2, \omega_1)}$ and $(*\omega_1, *\omega_2) = (\omega_1, \omega_2)$.

The smooth differential $\omega \in \mathcal{E}^1(X)$ is closed (resp., co-closed) if $d\omega = 0$ (resp., $d * \omega = 0$). The $\mathcal{E}^1(X)$ differential ω is exact (resp., co-exact) if $\omega = df$ (resp., $\omega = *df$), for $f \in \mathcal{E}(X) = C^\infty(X)$ smooth functions on X.

Next we introduce subspaces $d\mathcal{E}(X)$ and $*d\mathcal{E}(X)$ of exact and co-exact differentials

$$d\mathcal{E}(X) = \overline{\{df | f \in \mathcal{E}(X)\}},$$
$$*d\mathcal{E}(X) = \overline{\{*df | f \in \mathcal{E}(X)\}},$$

where bar denotes closure in $L^1_2(X)$. If X is non-compact then we shall take functions with compact support.

$d\mathcal{E}(X)$ and $*d\mathcal{E}(X)$ are orthogonal, because $(df, *dg) = \int_X df \wedge dg = \int_X d(fdg) = 0$ by Stokes theorem, since X has no boundary. Consider orthogonal complements $d\mathcal{E}(X)^\perp$, $*d\mathcal{E}(X)^\perp$ under the scalar product (10). We have

Lemma **9.1.** *Let $\alpha \in L_1^2(X)$ be of class $\mathcal{E}^1(X)$. Then $\alpha \in *d\mathcal{E}^\perp \Leftrightarrow d\alpha = 0$ and $\alpha \in *d\mathcal{E}^\perp \Leftrightarrow d * \alpha = 0$*

Proof. Lets do it for $\alpha \in B^\perp$, applying Stokes theorem again

$$(\alpha, df) = \overline{(df, \alpha)} = \overline{\int_X d\bar{f} \wedge *\alpha} = -\overline{\int_X \bar{f} \wedge d * \alpha} = 0,$$

for any $f \in \mathcal{E}(X)$. Hence $d * \alpha = 0$. □

Definition **9.2** *A differential α is harmonic if it is smooth and both closed and co-closed.*

It follows immediately from definition that locally, harmonic differentials have the form

$$\alpha = f(z)dz + \overline{g(z)}d\bar{z},$$

where f, g are holomorphic, and also $\alpha = dh$, where where h is harmonic function ($\partial\bar{\partial}h = 0$). The proof is straightforward and is left as an exercise. Then it follows that

$$\alpha + i * \alpha = 2fdz \tag{11}$$

is a holomorphic differential.

Consider now the space $H = d\mathcal{E}^\perp \cap *d\mathcal{E}^\perp$, which is intersection of orthogonal complements. All harmonic differentials by definition are in H. The stronger statement is that H consists only of harmonic differentials (for technical proof based on Weyl's lemma we refer to Jost [Theorem 5.2.1]).

Hence the statement is

Corollary **9.3.** *Every square-integrable smooth differential ω on X is represented by an orthogonal sum*

$$\omega = df + *dg + \alpha$$

of exact, co-exact and harmonic forms, i.e. $\mathcal{E}^1(X) = d\mathcal{E}(X) \oplus *d\mathcal{E}(X) \oplus H$

This discussion can be continued further to prove that $\dim H = 2g$, $\dim H^1(X, \mathbb{C}) = 2g$ and the dimension of space of holomorphic differentials is g, see [FarkasKra].

9.2. **Existence and uniqueness of differentials of 2nd and 3rd kind.**

Theorem **9.4.** *Given points P, Q on a compact Riemann surface X and a canonical basis of cycles there exists unique normalized Abelian differentials $\eta_P^{(N)}, N \in \mathbb{N}$ of 2nd kind and η_{PQ} of 3rd kind.*

Proof. Uniqueness of 2nd and 3rd kind differentials follows from simple considerations: the difference of two normalized differentials is a holomorphic differential with vanishing a-cycles. Hence it vanishes identically due to Cor. 8.1.

The existence can be verified by the following explicit construction. Consider nested neighbourhoods $P \in U_0 \subset U_1 \subset X$ and a $C^\infty(X)$ interpolating function $\rho : \rho = 1$ on U_0 and $\rho = 0$ on $X \backslash U_1$. Let z be a local coordinate on U_1 centered at P and consider the differential on $X \backslash \{P\}$

$$\psi = d\left(-\frac{\rho}{Nz^N}\right) = \left(-\frac{\partial\rho}{Nz^N} + \frac{\rho}{z^{N+1}}\right) dz - \left(\frac{\bar{\partial}\rho}{Nz^N}\right) d\bar{z}.$$

The $(0,1)$ part of ψ is smooth on X and following Cor. 9.3 it can be decomposed as

$$\psi - i * \psi = df + *dg + \alpha \tag{12}$$

into exact, co-exact and harmonic parts. Consider now differential $\gamma = \psi - df$.

The key claim is that γ is harmonic on $X \backslash P$ and $\gamma - \frac{dz}{z^{N+1}}$ is harmonic on U_0. Indeed,

$$\gamma = d\left(-\frac{\rho}{Nz^N} - f\right),$$

so it is closed on $X \backslash P$, and from Eq. (12) it follows that $\gamma = i * \psi + *dg + \alpha$, so it is co-closed. Hence $\gamma \in H(X \backslash P)$.

Next, observe that $\psi - \frac{dz}{z^{N+1}} \equiv 0$ on U_0 by construction. Hence,

$$\gamma - \frac{dz}{z^{N+1}} = -df = *dg + \alpha \quad \text{on } U_0,$$

so $\gamma - \frac{dz}{z^{N+1}} \in H(U_0)$.

Then the direct corollary of (11) is that the differential $\eta = \frac{1}{2}(\gamma + i * \gamma)$ is holomorphic on $X \backslash P$ and $\eta - \frac{dz}{z^{N+1}}$ is holomorphic on U_0. Hence η has exactly the pole of the order $N+1$ at P and holomorphic otherwise.

In order to prove the existence of the 3rd kind differential the above construction shall be applied to

$$\psi_{z_1 z_2} = d\left(\rho \log \frac{z - z_1}{z - z_2}\right),$$

for $z_1, z_2 \in U_0$. For arbitrary two points P, Q we can do a telescopic sum of $\psi_{z_1 z_2}$. □

9.3. **Abel theorem.** The Abel theorem describes what happens to the principal divisors under the Abel map, defined in the previous lecture.

Theorem **9.1.** **(Abel theorem)** *The divisor is principal if and only if* $\deg D = 0$ *and* $I(D) = 0 \mod \Lambda$.

Proof. Let f be a meromorphic function. As we have already shown in Lemma 8.10, $\deg(f) = 0$. Hence we can write for its divisor

$$(f) = P_1 + ... + P_N - Q_1 - ... - Q_N,$$

where some of the points could coincide. Consider the meromorphic differential

$$\eta = d \log f$$

Since f is s scalar function, the periods of η can only be integer multiples of $2\pi i$,

$$\int_{a_j} \eta = 2\pi i n_j, \quad \int_{b_j} \eta = 2\pi i m_j, \quad n_j, m_j \in \mathbb{Z}.$$

We need to compute $I(D)$. Following the definition $I(D)$, in sec. 8.5, we have

$$I((f)) = \sum_{k=1}^{N} \int_{Q_k}^{P_k} \omega_j.$$

In order to compute this we apply the Riemann bilinear identity Eq. (4), Thm. 8.1 to $\omega_1 = \omega_l$, where ω_l is the l-th element of canonical basis of holomorphic differentials, and to $\omega_2 = \eta$. Note that $\eta = d \log f$ is closed. We have

$$\begin{aligned}\frac{1}{2\pi i} \int_{\partial X_0} \left(\eta(P) \int_{P_0}^{P} \omega_l \right) &= \sum_{P \in X} \operatorname{Res} \eta(P) \int_{P_0}^{P} \omega_l = \sum_{k=1}^{N} \int_{Q_k}^{P_k} \omega_l \\ &= \frac{1}{2\pi i} \sum_{j=1}^{g} (2\pi i m_j \delta_{jl} - 2\pi i \tau_{jl} n_j) = m_l - \tau_{lj} n_j \in \Lambda,\end{aligned}$$

hence

$$I((f)) = 0 \mod \Lambda.$$

In the opposite direction, consider a divisor $D = P_1 + ... + P_N - Q_1 - ... - Q_N$ of degree zero, such that

$$I(D) = 0 \mod \Lambda. \tag{13}$$

The main idea is to construct the meromorphic function f with the divisor $(f) = D$.

Let us consider the normalized abelian differentials of the third kind $\eta_{P_k Q_k}$ and let

$$\tilde{\eta} = \sum_{k=1}^{N} \eta_{P_k Q_k}. \tag{14}$$

Then a-periods of $\tilde{\eta}$ vanish by definition. Consider b-periods of $\eta_{P_k Q_k}$ and apply Eq. (4) Thm. 8.1 for $\omega_1 = \omega_l$ (holomorphic differential from canonical basis) and $\omega_2 = \eta_{P_k Q_k}$. Then

$$\int_{b_l} \eta_{P_k Q_k} = \int_{\partial X_0} \left(\eta_{P_k Q_k}(P) \int_{P_0}^{P} \omega_l \right) = 2\pi i \left(\int_{P_0}^{P_k} \omega_l - \int_{P_0}^{Q_k} \omega_l \right) = 2\pi i \int_{Q_k}^{P_k} \omega_l.$$

Hence,

$$\int_{b_l} \tilde{\eta} = \sum_{k=1}^{N} \int_{b_l} \eta_{P_k Q_k} = 2\pi i \sum_{k=1}^{N} \int_{Q_k}^{P_k} \omega_l = 2\pi i I(D) = 2\pi i (n_l + \tau_{lj} m_j) \in \Lambda,$$

for some $n_l, m_l \in \mathbb{Z}$, according to the assumption (13). Then the function

$$f(P) = \exp\left(\int_{P_0}^{P} \tilde{\eta} - 2\pi i \sum_{j=1}^{g} m_j \int_{P_0}^{P} \omega_j \right),$$

where $\tilde{\eta}$ is given by Eq. (14) is single-valued on X and is a meromorphic function with the divisor D.

□