10. Theta functions

10.1. **Theta functions in one variable.** Theta function is an analytic function of $z \in \mathbb{C}$ is defined as

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z} \tag{1}$$

and parameter $\tau \in \mathbb{H}$ takes values in the upper-half plane, i.e., $\operatorname{Im}\tau > 0$. The series converges absolutely and uniformly on compact sets. Indeed for $|\operatorname{Im} z| < c$ and $\operatorname{Im}\tau > \epsilon$ we have

$$|e^{\pi i n^2\tau + 2\pi i n z}| < e^{-\pi\epsilon n^2 + 2\pi c n} < e^{-\pi\epsilon n(n-2c/\epsilon)}$$

hence starting from $n_0 > 2c/\epsilon$ the series begin to rapidly converge.

Theta function is almost periodic with respect to the lattice $\Lambda = m' + m\tau,\ m', m \in \mathbb{Z}$. Indeed,

$$\begin{aligned}
\vartheta(z+1,\tau) &= \vartheta(z,\tau),\\
\vartheta(z+\tau,\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n(z+\tau)}\\
&= \sum_{n\in\mathbb{Z}} e^{\pi i (n+1)^2\tau - \pi i\tau + 2\pi i n z} = e^{-\pi i\tau - 2\pi i z}\vartheta(z,\tau)
\end{aligned}$$

and in general

$$\vartheta(z+m',\tau) = \vartheta(z,\tau), \quad \vartheta(z+m\tau,\tau) = e^{-\pi i m^2\tau - 2\pi i m z}\vartheta(z,\tau), \quad m', m \in \mathbb{Z}.$$

Theta functions with characteristics are defined as follows

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\big(\pi i (n+a)^2\tau + 2\pi i (n+a)(z+b)\big) = \\ e^{\pi i a^2\tau + 2\pi i a(z+b)}\vartheta(z+a\tau+b,\tau), \tag{2}$$

for $a, b \in \mathbb{R}$.

Especially important are theta-functions with half-integer characteristics (Jacobi theta functions)

$$\begin{aligned}
\theta_1(z,\tau) &= -\vartheta\begin{bmatrix} \frac12 \\ \frac12 \end{bmatrix}(z,\tau),\\
\theta_2(z,\tau) &= \vartheta\begin{bmatrix} \frac12 \\ 0 \end{bmatrix}(z,\tau),\\
\theta_3(z,\tau) &= \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z,\tau) = \vartheta(z,\tau),\\
\theta_4(z,\tau) &= \vartheta\begin{bmatrix} 0 \\ \frac12 \end{bmatrix}(z,\tau).
\end{aligned}$$

Note that $\theta_1(-z) = -\theta_1(z)$ is odd and $\theta_{2,3,4}(-z) = \theta_{2,3,4}(z)$ are even.

These functions satisfy quadratic relations (see Mumford, vol. I).

10.2. **Zeroes.** Theta functions are multivalued on the torus $T_\tau = C/\Lambda$, but its zeroes are well-defined on the torus, as follows from formulas above. We can immediately show that theta function Eq.(1) has one zero in the torus. The number of zeroes is given by the integral

$$\#\text{ zeroes of }\vartheta = \frac{1}{2\pi i}\int_{4\text{ sides}} \frac{d}{dz}(\log f)dz = 1 \tag{3}$$

See Fig. 1. From definition (2) it immediately follows that theta function with characteristics also has one zero. Its location can be determined as follows. It is not hard to show that

$$\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](z,\tau) = -\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](-z,\tau), \tag{4}$$

hence it vanishes at $z = 0$. Therefore from (2) one first infers that $\vartheta(z,\tau)$ vanishes at $\frac{1}{2}\tau + \frac{1}{2}$, and next, $\vartheta\left[\begin{array}{c}a\\ b\end{array}\right](z,\tau)$ has zeros at

$$\Big(a + \frac{1}{2}\Big)\tau + \Big(b + \frac{1}{2}\Big) \qquad \text{mod }\Lambda.$$

10.3. **Meromorphic functions on the torus.** On $\mathbb{P}^1$ we can construct meromorphic functions as ratios

$$\prod_j \frac{z - a_j}{z - b_j} \tag{5}$$

On the torus theta functions give us several ways to construct meromorphic functions

- We can take quotients of θ itself

$$\prod_{j=1}^N \frac{\vartheta(z - a_j)}{\vartheta(z - b_j)} \tag{6}$$

  This is periodic provided $\sum a_j = \sum b_j$. Hence, for $N = 1$ we only get constant function as no meromorphic functions with only one simple pole exist on torus.
- One can take second logarithmic derivative

$$\frac{d^2}{dz^2}\log\theta_1(z,\tau) = -\wp(z) + \text{const}$$

  where $\wp(z)$ is Weierstrass $\wp$-function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0}\left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right).$$

  The constant is chosen such that the Laurent expansion of $\wp$ at zero has no constant term. This function has a pole of order 2 at $z = 0$ hence $\wp(z)$ can be written as a sum of constant function and, e.g., meromorphic function of the type (6) with $\theta_1(z)^2$ in the denominator. Indeed,

$$\frac{d^2}{dz^2}\log\theta_1(z,\tau) = \frac{\theta_3''(0)}{\theta_3(0)} - \frac{\theta_1'(0)^2}{\theta_3(0)^2}\frac{\theta_3(z)^2}{\theta_1(z)^2},$$

see Mumford, p.26, for the proof using quadratic relations.

10.4. **Riemann theta function.** Consider now coordinate vector $z_j \in \mathbb{C}^g$ and the lattice $\Lambda = m'_j + \tau_{jl} m_l,\ m, m' \in \mathbb{Z}^g$, where τ is symmetric complex matrix with positive definite imaginary part $\operatorname{Im} \tau > 0$.

Remark: We reduce to the tori of this form, called principally polarized Abelian tori, because its a classical theorem (Siegel) that non-constant meromorphic functions exist only on such Abelian tori. (Mumford, Griffits-Harris). Abelian tori in general correspond to the lattice $\Lambda = m'_j a_j + \tau_{jl} m_l,\ m, m' \in \mathbb{Z}^g, a_k \in \mathbb{N}, a_1 = 1, a_k | a_{k+1}$

Closely related fact is that for such tori there is projective embedding (Lefshetz embedding theorem).

This is parameterized by an open subset in $\mathbb{C}^{g(g+1)/2}$, called the Siegel upper-half plane. Then

$$\vartheta(z, \tau) = \sum_{n_j \in \mathbb{Z}^g} e^{\pi i n_j \tau_{jl} n_l + 2\pi i n_j z_j}, \tag{7}$$

where we drop the indices on z, τ etc., where notation is obvious.

Proposition **10.1.** *ϑ converges absolutely and uniformly in each set* $\max_j |\operatorname{Im} z_j| < c_1$ *and* $\operatorname{Im} \tau_{jl} \geqslant c_2 \delta_{jl}$.

Proof.

$$|e^{\pi i n_j \tau_{jl} n_l + 2\pi i n_j z_j}| \leqslant e^{-\pi c_2 \sum n_j^2 + 2\pi c_1 \sum_j |n_j|}. \tag{8}$$

Hence the series are dominated by $\left(\sum_{n \geqslant 0} e^{-\pi c_2 \sum n^2 + 2\pi c_1 |n|}\right)^g$, which we already know converges. □

Theta function is quasi-periodic

$$\vartheta(z_j + m'_j + \tau_{jl} m_l, \tau) = e^{-\pi i m_j \tau_{jl} m_l - 2\pi i m_j z_j} \vartheta(z, \tau), \quad m, m' \in \mathbb{Z}^g. \tag{9}$$

The proof is identical to the one-dimensional case.

11. Divisors and Jacobi inversion theorem

11.1. **Divisor classes.**

Definition **11.1** *Two divisors D, D' are called linearly equivalent with their difference $D - D' = (f)$ is a principal divisor (=divisor of a meromorphic function). All linearly equivalent divisors belong to the equivalence class, called divisor class, which is labelled as, e.g., $[D]$.*

Since any principal divisor has degree zero, all divisors in the same divisor class have the same degree. Notation for equivalent divisors: $D \equiv D'$.

Definition **11.2** *All Abelian differentials belong to the same divisor class, called the canonical class K.*

This is because the ratio ω_1/ω_2 of any two Abelian differentials is a meromorphic function.

It is an immediate consequence of the Abel theorem that Abel map depends only on the divisor class.

Corollary **11.3.** *All divisors in the same divisor class map to the same point in the Jacobian.*

Proof. From linearity of the Abel map it follows

$$I(D+(f)) = I(D) + I((f)) = I(D),$$

by Abel theorem. □

11.2. **Recap: Riemann-Roch theorem.**

Theorem **11.1.** *For a divisor D on a Riemann surface of genus g*

$$\dim H^0(X, \mathcal{O}(D)) = \deg D - g + 1 + \dim H^0(X, \Omega(-D)).$$

Here

$$H^0(X, \mathcal{O}(D)) = \{f \text{ meromorphic on } X \mid (f) + D \geqslant 0 \text{ or } f \equiv 0\}$$

and

$$H^0(X, \Omega(-D)) := \{\omega \text{ Abelian differential } X \mid (\omega) \geqslant D \text{ or } \omega \equiv 0\}.$$

Notations:

$$h^0(D) = \dim H^0(X, \mathcal{O}(D))$$

($\equiv l(-D)$ can be also encountered),

$$i(D) = \dim H^0(X, \Omega(-D)),$$

the latter is also called index of speciality. Hence

$$h^0(D) = \deg D - g + 1 + i(D).$$

Clearly, dimensions $h^0(D)$ and $i(D)$ depend only on the divisor class. If $D - D' = (\tilde{f})$ then vector spaces are identified by multiplication by $\tilde{f}$.

Lemma **11.4.**

$$i(D) = h^0(K - D) \tag{10}$$

Proof. Let ω_0 be an Abelian differential with divisor $(\omega_0) \in [K]$. Then the map $\omega \in H^0(\Omega(-D)) \ni \omega \to \omega/\omega_0 \in H^0(\mathcal{O}((\omega_0) - D)) = H^0(\mathcal{O}(K - D))$ is an isomorphism of of vector spaces, hence their dimensions are equal. □

This is a consequence of a more profound isomorphism between the corresponding vector spaces, called Serre duality.

11.3. **Canonical class.**

COROLLARY **11.5.** *The degree of the canonical class* $\deg K = 2g - 2$.

Proof. On the sphere dz has double pole at infinity. $dz = -dw/w^2, z = 1/w$. For $g > 0$ the Rieman-Roch theorem states that $h^0(K) = \deg K - g + 1 + i(K)$. Form Eq. (10) it follows that

$$i(K) = h^0(0) = 1, \quad \text{and} \quad h^0(K) = i(0) = g,$$

the latter is because there are g independent holomorphic differentials. Hence $\deg K = 2g - 2$. □

COROLLARY **11.6.** *There is no point on X where all holomorphic differentials vanish simultaneously.*

Proof. Suppose there exists such a point $P \in X$. Then for divisor $D = P$, we have $i(P) = g$ and from the Riemann-Roch theorem it follows $h^0(D) = 2$. Then besides the constant function, there exists a nontrivial meromorphic function with only one simple pole at P. Due to the Ex. 2, Homework 2 X is biholomorphic to the sphere. □

11.4. **The Abel map as an embedding.**

DEFINITION **11.7** *A holomorphic map $F : X \to Y$ between complex manifolds is called embedding if F is an immersion (derivative is injective at every point) and $F : X \to F(X)$ is an homeomorphism.*

LEMMA **11.8.** *If X is compact, this is equivalent to F being injective immersion.*

Proof. Indeed, then $F : X \to F(X)$ is bijective and continuous. We will use the following fact: a function g is continuous iff $g^{-1}(C)$ is closed for all C closed in X. Take $g = F^{-1}$, then $g^{-1}(C) = F^{-1^{-1}}(C) = F(C)$. C closed in X compact means C is compact. Since F is continuous, it follows that $F(C)$ is compact. Hence $F(C)$ is closed. Hence F^{-1} is continuous. Hence $F : X \to F(X)$ is a homeomorphism. □

LEMMA **11.9.** *The Abel map* $I(P) = \int_{P_0}^{P} \omega_j$ *is an embedding.*

Proof. Derivative of the Abel map at a point P equals

$$dI(P) = \omega_j(P)$$

From Cor. 11.6 we know that for any point $P \in X$ holomorphic differentials cannot vanish at P, hence $dI(P) \neq 0$, so the Abel map is an immersion.

Suppose that two point $P_1, P_1 \in X$ have the same image $I(P_1) = I(P_2)$. Then $I(P_1 - P_2) \equiv 0$ and by Abel theorem $P_1 - P_2$ is a principal divisor. By Ex. 2, Homework 2 meromorphic function with just one simple pole does not exist for $g > 0$, hence $P_1 = P_2$. □

11.5. **Jacobi inversion theorem.** The set X_n of positive divisors of degree n can be described as nth symmetric product of X with itself, $X_n = X \times ... \times X/Sym_n$, where quotient by the symmetric group Sym_n means that we do not distinguish between the points.

In what follows we will also need a notion of a special divisor.

Definition **11.10** *A positive divisor D of degree g is called special if $i(D) > 0$.*

(Hence the name index of speciality). In other words there exists a non-zero holomorphic differential ω with divisor

$$(\omega) \geqslant D. \tag{11}$$

This is rare. Indeed, since holomorphic differentials form a dim-g vector space, we can write $\omega = \sum \alpha_j \omega_j$ for some basis. Then Eq. (11) translates into a homogeneous system of linear equations on coefficients α_j, one equation for each zero of $D = P_1 + ... + P_g$,

$$\sum \alpha_j \omega_j(P_k) = 0.$$

So to get non-zero α_j's we need the condition $\det \omega_j(P_k) = 0$. Hence most positive divisors are non-special. In particular, in the proof of next theorem we will see that in for every non-special divisor there is a neighbourhood where all divisors are also non-special.

Theorem **11.2. (Jacobi inversion theorem)** *Consider the set X_g of positive divisors of degree g. The Abel map*

$$I: \ X_g \to Jac(X)$$

on this set is surjective.

Proof. We should show that for any point $C_j \in Jac(X)$ there exists a positive divisor $D = P_1 + ... + P_g$ of degree g, such that

$$C_j = \sum_{l=1}^{g} \int_{P_0}^{P_l} \omega_j.$$

Let us start with some non-special degree-g divisor $D_z = z_1 + ... + z_g$. Consider the Abel map for this divisor $I(D_z)$ and compute its differential

$$d_{z_l} I(D_z) = \omega_j(z_l).$$

Hence the Jacobian matrix of the map is

$$\begin{pmatrix} \omega_1(z_1) & \dots & \omega_1(z_g) \\ \vdots & & \vdots \\ \omega_g(z_1) & \dots & \omega_g(z_g) \end{pmatrix} \tag{12}$$

By the assumption that D_z is a non-special divisor, the determinant of this matrix is non-zero.

Choose z_1 such that $\omega_1(z_1) \neq 0$. By subtracting first row we can set the following entries to zero: $\omega_2(z_1) = ... = \omega_g(z_1) = 0$. Choose then z_2 such that $\omega_2(z_2) \neq 0$, and repeat the procedure. Finally we will get upper-triangular matrix with non-zero entries on the diagonal, hence it is not-degenerate. This holds also in the neighbourhood of $z_1 + ... + z_g$.

One can show that all divisors in a neighbourhood of D_z are non-special. Therefore, by the implicit function theorem I maps the neighbourhood of $(z_1, ..., z_g)$ bijectively onto a neighbourhood $V_{I(D_z)} \subset Jac(X)$ of the point $I(D_z) \in Jac(X)$.

Now let $C_j \in Jac(X)$ be an arbitrary point. One can always find $n \in \mathbb{N}$ big enough so that

$$I(D_z) + \frac{1}{n}C \in V_{I(D_z)}$$

Then there exists another non-special divisor D_y (in the vicinity of D_z) such that it is the preimage of the point above.

$$I(D_y) = I(D_z) + \frac{1}{n}C.$$

Then

$$C = n(I(D_y) - I(D_z))$$

and we need to show that

$$C = I(D), \quad \text{where } D = P_1 + ... + P_g \text{ is a positive divisor of } \deg g.$$

Consider the divisor

$$D' = n\sum_{j=1}^{g} y_j - n\sum_{j=1}^{g} z_j + gP_0$$

of degree g. By Riemann-Roch theorem,

$$h^0(D') = g + 1 - g + i(D') \geqslant 1.$$

Hence there exists a meromorphic function f with divisor $(f) + D' \geqslant 0$. Hence $(f) + D'$ is a positive divisor of degree g and we can write for this divisor

$$D = P_1 + ... + P_g = (f) + D'$$

Applying Abel theorem for this divisor we get

$$I(D) = I((f) + nD_y - nD_z + gP_0) = I((f)) + I(nD_y - nD_z) + I(gP_0) = 0 + C + 0.$$

$\square$

12. Theta divisor

12.1. **Zeroes of the Riemann theta function.** Obviously $\vartheta(e) \not\equiv 0$ for $e \in \mathbb{C}^g$ because it is given by a Fourier expansion with non-zero coefficients. Let now τ_{jl} be a period matrix of the Riemann surface X of genus g, so we restrict to the p.p. abelian varieties which are a Jacobian of a Riemann surface.

The set of zeroes of theta function is called *theta divisor*. The goal of this lecture is to describe the theta divisor in terms of divisors on X.

The function theory on X can be studied using the Jacobean embedding, via theta function

$$f(P) = \vartheta\left(\int_{P_0}^{P} \omega_j - e_j, \tau\right) \tag{13}$$

as a function of a point $P \in X$, for an arbitrary vector $e_j \in \mathbb{C}^g$.

This function is locally single-valued, but globally multi-valued on X. It is invariant around a-cycles. Around b-cycles it transforms as

$$\vartheta\left(-e_j + \int_{P_0}^{P} \omega_j + \int_{b_k} \omega_j, \tau\right) = \vartheta\left(-e_j + \int_{P_0}^{P} \omega_j + \tau_{kj}, \tau\right)$$
$$= e^{-\pi i \tau_{kk} - 2\pi i (\int_{P_0}^{P} \omega_k - e_k)} \vartheta\left(-e_j + \int_{P_0}^{P} \omega_j, \tau\right). \tag{14}$$

It follows that its zeroes are well-defined on X.

Theorem **12.1. (Riemann vanishing theorem)**.

(1) Theta function either vanishes identically $f(P) \equiv 0$ on X or has g zeroes (counting multiplicities) $Q_1, ..., Q_g$.

(2) In the latter case there exists a vector $\Delta_j \in \mathbb{C}^g$, such that

$$\sum_{l=1}^{g} \int_{P_0}^{Q_l} \omega_j = e_j - \Delta_j \mod \Lambda. \tag{15}$$

Proof. (1) Consider the canonical dissection X_0 and assume all zeroes are separate and $Q_i \in X_0$, $P_0 \in X_0$. Let δ_j be small disks around Q_j. Consider differential df/f and apply Stokes theorem

$$0 = \int_{X_0 - \cup\delta_j} d\frac{df}{f} = \int_{\partial(X_0 - \cup\delta_j)} \frac{df}{f} = -\sum_j \int_{\partial\delta_j} \frac{df}{f}$$
$$+ \sum_{l=1}^{g} \left(\int_{a_l^-} - \int_{a_l^+} + \int_{b_l^-} - \int_{b_l^+}\right) d\log f, \tag{16}$$

since df/f is holomorphic in $X_0 - \cup\delta_j$. Since f is invariant under a-cycles, b-integrals cancel out. Since b_l joins a_l^- and a_l^+ and around b_l-cycle we have Eq. (14) $d\log f|_{a_l^-} - d\log f|_{a_l^+} = 2\pi i\omega_l$. Hence

$$\#\text{ of zeroes of } f = \frac{1}{2\pi i}\sum_j \int_{\partial\delta_j} \frac{df}{f} = \sum_{l=1}^{g} \int_{a_l} \omega_l = g. \tag{17}$$

(2) Here the goal is to derive the formula for the vector of Riemann constants Δ_j

$$\Delta_j = \frac{1}{2} + \frac{1}{2}\tau_{jj} - \sum_{l\neq j} \int_{a_l} \omega_l \int_{P_0}^{P} \omega_j \tag{18}$$

and the idea is to apply the previous argument to the one form $g_k df/f$, where $\omega_k = dg_k$ and $g_k(P_0) = 0$ on X_0.

$$0 = \int_{X_0 - \cup\delta_j} d\big(g_k \frac{df}{f}\big) = ... \tag{19}$$

see [Mumord, vol. I, p.150]. One can check that Δ_j in Eq. (18) is independent of the integration path, but depends on the base point.

□