Riemann Surfaces - Homework 1

1. Problem

Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Gamma' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ be two lattices in $\mathbb{C}$, i.e. $\omega_j, \omega_j' \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for $j \in 1, 2$ and $\operatorname{Im} \dfrac{\omega_1}{\omega_2} \neq 0$ resp. $\operatorname{Im} \dfrac{\omega_1'}{\omega_2'} \neq 0$. Show that $\Gamma = \Gamma'$ if and only if

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$$

for some matrix $A \in \mathrm{SL}(2, \mathbb{Z}) := \{A \in \mathrm{GL}(2, \mathbb{Z}) \mid \det A = 1\}$.

2. Problem

(a) Let $\Gamma, \Gamma' \subset \mathbb{C}$ be two lattices and $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$. Show that the map

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \alpha z \end{aligned}$$

induces a holomorphic map $\tilde{f} : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ which satisfies

$$\tilde{f} \circ \pi = \pi' \circ f$$

where $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ resp. $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ are the standard projection maps. Furthermore, show that $\tilde{f}$ is biholomorphic if and only if $\alpha\Gamma = \Gamma'$.

(b) Show that every torus $X = \mathbb{C}/\Gamma$ (seen as additive group) is isomorphic to a torus of the form

$$X(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

where $\tau \in \mathbb{C}$ satisfies $\operatorname{Im}(\tau) > 0$.

(c) Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and $\tau \in \mathbb{C}$ such that $\operatorname{Im}(\tau) > 0$. Let

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Show that the tori $X(\tau)$ and $X(\tau')$ are isomorphic.

3. Problem

Show that every automorphism of $\mathbb{C}$ is of the form $\Phi(z) = az + b$, where $a \neq 0$.

Hint. Show that Φ has a pole of order one at ∞ by using the Casoratti-Weierstrass and open mapping theorems.

4. Problem

Any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$ defines a Möbius transformation

$$M_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \;\; M_A(z) = \frac{az + b}{cz + d}.$$

Here we set $M_A(-d/c) = \infty$ and $M_A(\infty) = a/c$

(a) Show that $M_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an automorphism and find $(M_A)^{-1}$.

(b) Show that the set $\mathcal{M}$ of all Möbius transformations forms a group for the composition of maps.

(c) Show that $\mathrm{GL}(2,\mathbb{C}) \to \mathcal{M}$, $A \mapsto M_A$ is a group morphism whose kernel is $\mathbb{C}I_2$ and this morphism induces an isomorphism $\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\{I_2, -I_2\} \cong \mathcal{M}$.

(d) Show that if ∞ is a fixed point of $\Phi \in \operatorname{Aut}\widehat{\mathbb{C}}$, then $\Phi(z) = az + b$, where $a \neq 0$.

(e) Show that $\operatorname{Aut}\widehat{\mathbb{C}}$ acts transitively on $\widehat{\mathbb{C}}$, that is, for any two points $z_1, z_2 \in \widehat{\mathbb{C}}$ there exists $\Phi \in \operatorname{Aut}\widehat{\mathbb{C}}$ with $\phi(z_1) = z_2$.

(f) Show that $\operatorname{Aut}\widehat{\mathbb{C}} = \mathcal{M}$.

5. Problem

Let $\Gamma \subset \mathbb{C}$ be a lattice. The Weierstass $\wp$ function with respect to the lattice Γ is defined by

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

(a) Show that $\wp_\Gamma$ is a doubly periodic meromorphic function with poles in the points of Γ. (A meromorphic function F on $\mathbb{C}$ is called doubly periodic if for all $z \in \mathbb{C}$, $\omega \in \Gamma$, we have $F(z+\omega) = F(z)$.)

(b) Show that $\wp_\Gamma$ defines a meromorphic function on the torus $\mathbb{C}/\Gamma$.

(c) Let f be a doubly periodic meromorphic function on $\mathbb{C}$ with poles in the points of Γ, whose Laurent expansion at 0 is $\frac{1}{z^2} + \sum_{n\geq 1} a_n z^n$. Show that $f = \wp_\Gamma$.

6. Problem

Let $\Gamma \subset \mathbb{C}$ be a lattice and $\wp$ be the The Weierstass $\wp$ function with respect to the lattice Γ.

(a) Let f be a doubly periodic meromorphic function f. Show that there exists a polynomial $P \in \mathbb{C}[z]$ such that $P(\wp)f$ is a doubly periodic meromorphic function with poles in Γ.

Hint. For any zero a of f consider $(\wp(z) - \wp(a))^n$ for n large enough.

(b) Let f be a doubly periodic meromorphic function f with poles in Γ. Assume that f is even, that is, $f(z) = f(-z)$. Show that there exists a polynomial $P \in \mathbb{C}[z]$ such that $f = P(\wp)$.

(c) Let f be a doubly periodic meromorphic function f. Show that there exist rational functions $r, q \in \mathbb{C}(z)$ such that $f = r(\wp) + q(\wp)\wp'$.

Hint. The quotient of two even functions is even.

(d) Show that $z \in \mathbb{C}$ is a zero of of $\wp'$ if and only if $z \notin \Gamma$ and $2z \in \Gamma$. Conclude that $\wp'$ has exactly three zeros of order one on the torus $\mathbb{C}/\Gamma$.

(d) Let ω_1, ω_2 be a basis of Γ, that is, $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Show that

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),$$

where $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, $e_3 = \wp((\omega_1 + \omega_2)/2)$. Show that e_1, e_2, e_3 depend only on Γ and not on the choice of ω_1, ω_2.

(e) Show that the field $\mathcal{M}(\mathbb{C}/\Gamma)$ of meromorphic functions on the torus $\mathbb{C}/\Gamma$ is isomorphic to $\mathbb{C}(X)[Y]/(Y^2 - 4(X-e_1)(X-e_2)(X-e_3))$.

Hint. Consider the morphism $\mathbb{C}(X)[Y] \to \mathbb{C}/\Gamma$, $(X,Y) \mapsto (\wp, \wp')$.