Riemann Surfaces - Homework 2

1. Problem

Show that:

(a) Two meromorphic functions on a compact Riemann surface having the same principal part at each of their poles must differ by a constant.

(b) Two meromorphic functions on a compact Riemann surface having the same zeroes and poles (multiplicities included) agree up to a constant factor.

2. Problem

Let X be a compact Riemann surface. Prove that if there is a meromorphic function on X having exactly one pole, and that pole has order one, then X is biholomorphic to the Riemann sphere.

3. Problem

Determine the ramification points of the map $f : \mathbb{C} \to \mathbb{P}^1$, $f(z) = \frac{1}{2}(z + \frac{1}{z})$.

4. Problem

(a) Show that $\tan : \mathbb{C} \to \mathbb{P}^1$ is a local homeomorphism.

(b) Show that $\tan(\mathbb{C}) = \mathbb{P}^1 \setminus \{\pm i\}$ and that $\tan : \mathbb{C} \to \mathbb{P}^1 \setminus \{\pm i\}$ is a covering map.

(c) Let $X := \mathbb{C} \setminus \{it : t \in \mathbb{R}, |t| > 1\}$. Show that for every $k \in \mathbb{Z}$ there exists a unique holomorphic function $\arctan_k : X \to \mathbb{C}$ with $\tan \circ \arctan_k = \mathrm{id}_X$ and $\arctan_k(0) = k\pi$ (the kth branch of $\arctan$).

5. Problem

Let $P \in \mathbb{C}[z, w]$ be an irreducible polynomial

$$P(z, w) = w^n + p_{n-1}(z)w^{n-1} + \ldots + p_1(z)w + p_0(z)\,,$$

such that $(\partial P/\partial z, \partial P/\partial w) \neq (0, 0)$. Consider the Riemann surface $X = \{P(z, w) = 0\}$. Show that the projection π of X to the z-plane is a proper holomorphic map and calculate its degree.

6. Problem

Let $\Gamma \in \mathbb{C}$ be a lattice and let $\wp = \wp_\Gamma$ be the associated Weierstrass $\wp$-function. Let e_1, e_2, e_3 be defined as in Problem 6 of sheet 1.

(a) Show that $X = \big\{[z_0 : z_1 : z_2] \in \mathbb{P}^2 : z_0 z_2^2 = 4(z_1 - e_1 z_0)(z_1 - e_2 z_0)(z_1 - e_3 z_0)\big\}$ is a compact Riemann surface.

(b) Show that the image of the map

$$\Phi : \mathbb{C}/\Gamma \longrightarrow \mathbb{P}^2, \quad z + \Gamma \longmapsto \begin{cases} [1 : \wp(z) : \wp'(z)], & z \in \mathbb{C} \setminus \Gamma, \\ [0 : 0 : 1], & z \in \Gamma, \end{cases}$$

is X and that $\Phi : \mathbb{C}/\Gamma \to X$ is a biholomorphic map.

7. Problem

(a) Let $X = \{(z,w) \in \mathbb{C}^2 : z^3 + w^3 = 1\}$. Show that X is a Riemann surface and exhibit an atlas. Show that the projections $p_1, p_2 : X \to \mathbb{C}$, $p_1(z,w) = z$, $p_2(z,w) = w$ are holomorphic maps. Find the ramification points of p_1.

(b) Let $Y = \{[z : w : u] \in \mathbb{P}^2 : z^3 + w^3 = u^3\}$. Show that Y is a Riemann surface and the map $\varphi([z : w : u]) = w/u$ is a meromorphic function on Y. Find the poles of φ.

8. Problem

Show that a complex algebraic curve in $\mathbb{C}^2$ cannot be compact and cannot have isolated points.

9. Problem

Show that the complex line in $\mathbb{P}^2$ through the points $[0,1,1]$ and $[t,0,1]$ meets the projective curve $X = \{z_0^2 + z_1^2 = z_2^2\}$ in the two points $[0,1,1], [2t, t^2-1, t^2+1]$. Show that there exist a biholomorphism of Riemann surfaces between the complex projective line defined by $z_1 = 0$ and X given by $[1,0,0] \to [0,1,1]$ and $[t,0,1] \to [2t, t^2-1, t^2+1]$. Deduce that the complex solutions of Pythagoras' equation $z_0^2 + z_1^2 = z_2^2$ are

$$z_0 = 2\lambda\mu, \quad z_1 = \lambda^2 - \mu^2, \quad z_2 = \lambda^2 + \mu^2, \qquad \lambda, \mu \in \mathbb{C}.$$

What are the real and integer solutions?

10. Problem

Let $P \in \mathbb{C}[z,w]$ be a polynomial without multiple factors. The multiplicity of a curve $V(P) \subset \mathbb{C}^2$ defined by P at a point (z_0, w_0) is the smallest integer m such that $\partial_z^l \partial_w^j P(z_0,w_0) \neq 0$ for some $l, j \geq 0$, $l + j = m$. The polynomial

$$\sum_{l+j=m} \partial_z^l \partial_w^j P(z_0,w_0)(z-z_0)^l (w-w_0)^j \tag{1}$$

is homogeneous of degree m, so it can be factored as the product of m linear polynomials of the form $\alpha(z-z_0) + \beta(w-w_0)$. The lines defined by these linear polynomials are called the tangent lines to $V(P)$ at (z_0,w_0).

A point $(z_0,w_0) \in V(P)$ is called simple (resp. double, triple etc.) point if its multiplicity is one (resp. two, three etc.).

a) Show that a point $(z_0,w_0) \in V(P)$ is non-singular if and only if its multiplicity is one.

b) A singular point is called ordinary if the polynomial (**??**) has no multiple factors. Find the singular points of the curves defined by the following polynomials and decide if they are ordinary or not:

$$z^2 - w^2 - w^3, z^2 - w^3, (z^4 + w^4)^2 - z^2 w^2, (z^4 + w^4 - z^2 - w^2)^2 - 9z^2 w^2.$$

c) Show that $(z_0,w_0) \in V(P)$ is an ordinary double point if and only if $\partial_z P(z_0,w_0) = \partial_w P(z_0,w_0) = 0$ and $\partial_z \partial_w P(z_0,w_0) \neq \partial_z^2 P(z_0,w_0) \cdot \partial_w^2 P(z_0,w_0)$. An ordinary double point is also called node; a double point which is not ordinary is called cusp.