Riemann Surfaces - Homework 3

1. Problem

Consider the algebraic curve $X = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^3 - z_2^4 = 0\}$.

a) Show that X is a topological manifold, actually X is homeomorphic to $\mathbb{C}$.

b) Show that X is not a complex submanifold of $\mathbb{C}^2$.

2. Problem

A projective transformation of $\mathbb{P}^n$ is a bijection $f : \mathbb{P}^n \to \mathbb{P}^n$ such that for some linear isomorphism $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ we have $f([z]) = [T(z)]$, $z \in \mathbb{C}^{n+1} \setminus \{0\}$.

a) Show that a projective transformation is continuous.

b) Given $n+2$ distinct points $p_0, \dots, p_n, \xi \in \mathbb{P}^n$, no $n+1$ of each lie on a hyperplane, there is a unique projective transformation taking p_j to $[0, \dots, 0, 1, 0, \dots, 0]$ where 1 is on the jth place, and taking ξ to $[1, \dots, 1]$.

3. Problem

Let X be a plane algebraic curve of degree n. Show that it is possible to choose a coordinate system in $\mathbb{P}^2$ such that X possesses an affine equation of the following form:

$$f(z, w) = w^n + a_{n-1}(z) w^{n-1} + \dots + a_1(z) w + a_0(z)\,,$$

where $a_j \in \mathbb{C}[z]$, $\deg a_j \leqslant n - j$.

4. Problem

Let X be a nonsingular projective curve in $\mathbb{P}^2$ defined by a homogeneous polynomial $P \in \mathbb{C}[z_0, z_1, z_2]$ of degree $d > 1$. W.l.o.g. we can assume that $[0, 1, 0] \notin X$ after a projective transformation. Consider the map $p : X \to \mathbb{P}^1$, $p([z_0, z_1, z_2]) = [z_0, z_2]$. Show that:

(a) p is holomorphic.

(b) $[z_0, z_1, z_2] \in X$ is a ramification point of p if and only if

$$P(z_0, z_1, z_2) = \partial_2 P(z_0, z_1, z_2) = 0.$$

(c) $\nu_p([z_0, z_1, z_2])$ is the order of the polynomial $P(z_0, z, z_2)$ in z at $z = z_1$.

5. Problem

Consider a triangulation of a surface X with F faces, E edges and V vertices. The Euler characteristic of X is $\chi = \chi(X) = V - E + F$. Show that:

a) $2E = 3F$ and $E = 3(V - \chi)$,

b) $V \geq \dfrac{7 + \sqrt{49 - 24\chi}}{2}$,

c) A triangulation of the torus has at least 7 vertices. What is the minimal V for the sphere S^2?

6. Problem

a) Show that the word $\alpha_1\alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1}\alpha_3^{-1}$ represents a torus T^2.

b) The surface X is represented by the word $AabBabC$ for some words A, B, C. Show that X is also represented by the word $AuBuC$.

7. Problem

(i) Consider $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f(z) = z^3/(1-z^2)$. Show that f has degree 3 and find all its ramification and branching points. Verify Riemann-Hurwitz formula for f.

(ii) Consider $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f(z) = 4z^2(z-1)^2/(2z-1)^2$. Show that f has degree 4 and find all its ramification and branching points. Verify the Riemann-Hurwitz formula for f.

8. Problem

Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces.

(i) Show that if $Y \cong \mathbb{P}^1$, and f has degree at least two, then f must be ramified.

(ii) Show that if $g(X) = g(Y) = 1$, then f is unramified.

(iii) Show that $g(X) \geqslant g(Y)$ always.

(iv) Show that if $g(X) = g(Y) \geqslant 2$, then f is biholomorphic.

9. Problem

(a) Consider the algebraic curve $z^n + w^n = 1$ in $\mathbb{C}^2$. Find its projective closure X and its points at infinity. Show that X is a Riemann surface and calculate its genus.

(b) The same problem for the algebraic curve $w^2 = (z-\alpha_1)(z-\alpha_2)(z-\alpha_2)$, where α_1, α_2, α_3 are distinct complex numbers.

10. Problem

Let $\alpha_1, \ldots, \alpha_n$ be distinct complex numbers. Construct the projective closure of the algebraic curve given by $w^2 = (z-\alpha_1)\ldots(z-\alpha_n)$ and calculate its genus.