**Riemann Surfaces - Homework 4**

**1. Problem**
Let $(U, \varphi = z = x + iy)$ and $(V, \psi = w = u + iv)$ be two charts on a Riemann surface $X$.
(a) Show that

$$du \wedge dv = \frac{\partial(u,v)}{\partial(x,y)}\, dx \wedge dy = \left|\frac{\partial(\psi \circ \varphi^{-1})}{\partial z}\right|^2 dx \wedge dy,$$

where $\dfrac{\partial(u,v)}{\partial(x,y)} := \det J_{\psi\circ\varphi^{-1}}$ is the Jacobi determinant of $\psi \circ \varphi^{-1}$.
(b) Let $\alpha$ be a smooth form on $X$ such that $\alpha = f dx + g dy$ on $U$ and $\alpha = h du + j dv$ on $V$. Show that

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \left(\frac{\partial j}{\partial u} - \frac{\partial h}{\partial v}\right) du \wedge dv \quad \text{on } U \cap V,$$

hence $d\alpha := \left(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right) dx \wedge dy$ is a well defined $2$-form on $X$.

**2. Problem**
Suppose $X$ is a compact Riemann surface and $D$ is a divisor on $X$ with $\deg D < 0$. Show that $H^0(X, \mathcal{O}_D) = 0$.

**3. Problem**
Prove that the positions and orders of zeroes and poles of a meromorphic differential are well-defined, i.e., independent of the choice of charts and local holomorphic coordinate.

**4. Problem**
Recall that Jacobi theta function is defined as $\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$ for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, i.e., in the complex upper-half plane $\operatorname{Im}\tau > 0$.
(a) Find the zeroes of $\vartheta(z,\tau)$ in $z$. Then construct meromorphic functions on the torus $X = \mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$, for a divisor $D$ (check Abel's theorem). Is there a meromorphic function on $X$ with only one simple pole?
(b) Construct the differentials of the first, second and third kind on the torus.
(c) Fix a positive integer $n$, and choose any two sets of $n$ complex numbers $\{a_i : 1 \le i \le n\}$ and $\{b_j : 1 \le j \le n\}$ such that $\sum_i a_i - \sum_j b_j$ is an integer. Show that the ratio of translated theta functions

$$R(z) = \frac{\prod_{i=1}^n \theta\big(z - a_i, \tau\big)}{\prod_{j=1}^n \theta\big(z - b_j, \tau\big)}$$

is a meromorphic $\Gamma$-periodic function on $\mathbb{C}$ and so descends to a meromorphic function on $X$.
(d) Show that any meromorphic function on $X$ has the above form.

**5. Problem**
Consider the hyperelliptic Riemann surface $w^2 = \prod_{j=1}^N (z - z_j)$, where $z_j \neq z_k$ for $j \neq k$ and $N = 2g+1$ or $N = 2g+2$. Show that the differentials $z^{a-1}dz/w$, $a = 1, \ldots, g$, form a basis of holomorphic differentials.

**6. Problem**

(a) Let $p, q$ two disjoint points on the Riemann sphere $\mathbb{P}^1$. Find a meromorphic function $f$ on $\mathbb{P}^1$ such that $(f) = p - q$.

(b) Show that two divisors $D$ and $D'$ on $\mathbb{P}^1$ are linearly equivalent if and only if $\deg D = \deg D'$.

(b) Consider the divisor $D = 0 + 1$ on $\mathbb{P}^1$. Calculate $H^0(\mathbb{P}^1, \mathcal{O}(D))$.

**7. Problem**

Let $X$ be a complex torus and $p$ a point of $X$. Show that

$$\dim H^0(X, \mathcal{O}(np)) = \begin{cases} 0, & \text{for } n < 0, \\ 1, & \text{for } n = 0, \\ n, & \text{for } n \geq 1. \end{cases}$$

**8. Problem**

Let $X = \mathbb{C}/\Gamma$ be a complex torus, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$. Let $\wp(z, \tau)$ be the associated Weierstrass $\wp$-function.

(a) Calculate the divisors $(\wp)$ and $(\wp')$.

(b) Construct explicitely a meromorphic function on $X$ with an unique pole at a given point $p$ such that the order of this pole is a given integer $n \in \mathbb{N}$. Deduce that $\dim H^0(X, \mathcal{O}(np)) = n$.

(c) Calculate a canonical divisor of $X$.

**9. Problem**

Let $X$ be a compact Riemann surface of genus $g$.

(a) Show that for any point $p \in X$ there exists a meromorphic function on $X$, which has a pole at $p$ of order $\leq g + 1$ and is holomorphic on $X \setminus \{p\}$. Show that if $g = 1$ the order of the pole cannot be $1$.

(b) Show that there exists a branched covering $f : X \to \mathbb{P}^1$ with at most $g + 1$ sheets. Deduce that a compact Riemann surface of genus $g = 0$ is biholomorphic to $\mathbb{P}^1$.

(c) Show that for any $n \geq 2g$ there exists a meromorphic function with a unique pole and this pole has order $n$.

(d) Let $f : X \to X$ a biholomorphic map different from the identity. Show that $f$ has at most $2g + 2$ fixed points.

**10. Problem**

Let $X = \mathbb{C}/\Gamma$ be a complex torus, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$. Theta functions with characteristics $\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](z, \tau)$ were defined in the lecture. Prove the following Fay's identity:

$$\det \psi_i(z_j) = \mu(\tau)\vartheta\left[\begin{array}{c} \alpha + (N-1)/2 \\ \beta + (N-1)/2 \end{array}\right]\Big(\sum_1^N z_j, \tau\Big) \prod_{i<j} \vartheta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](z_i - z_j, \tau)$$

where $\psi_j = \vartheta\left[\begin{array}{c} \frac{\alpha+j}{N} \\ \beta \end{array}\right](Nz, N\tau)$, where $\mu(\tau)$ is independent of $z$.