Canonical Kähler metrics and quantisation

Joel Fine

Summer school in Cologne,
July 2012.

Caveat Lector. These notes have been written in a hurry, which has had two undesirable side effects. Firstly, I make no claim that the formulae are correct. Signs, factors of 2π and of i may be wrong. But I believe the contents to be true in spirit!. Secondly I have not had time to go through and add the extensive bibliography that the subjects covered deserve. There is, at the moment, not one single direct citation in the whole set of notes. Where the result is significant enough to accredit and I have been able to remember who to credit it to, I have done so by adding their name to it. When I have more time, I will hopefully add an extensive bibliography.

Contents

1 Brief review of Kähler basics

1.1 Chern connections and Chern classes

Definition 1.1. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle. A connection ∇ in E is said to be compatible with the holomorphic structure in E if $\pi^{0,1}(\nabla s) = \bar{\partial} s$ for all sections s of E.

Proposition 1.2. *Let E be a Hermitian holomorphic vector bundle. Then there is a unique connection in E compatible with both the Hermitian and holomorphic structures.*

Definition 1.3. The distinguished connection in the previous result is called the *Chern connection*.

We prove this for a line bundle $L \to X$. (The proof for vector bundles of higher rank is left as an exercise.) In a local holomorphic trivialisation, connections compatible with the holomorphic structure have the form $\nabla^A = \mathrm{d} + A$ where A is a $(1,0)$-form. Meanwhile, the Hermitian structure h is given in the trivialisation by a smooth real-valued positive function, which we continue to denote h. The condition $\nabla^A h = 0$ amounts to $Ah + h\bar{A} = \mathrm{d}h$ which, when combined with the fact that A is of type $(1,0)$, gives $A = \partial \log h$. It follows that there is a unique choice of A such that ∇^A is compatible with both structures. We can do this in each local trivialisation of L; by uniqueness the a priori locally defined Chern connections all agree over intersections and so give a globally defined connection.

Notice that the curvature of L in the local trivialisation is given by $dA = \bar{\partial}\partial \log h$. Write $h' = e^f h$ for a second Hermitian metric in L, where f is any smooth function $X \to \mathbb{R}$. The corresponding curvatures are related by $F_{h'} = F_h + \bar{\partial}\partial f$. It follows that the cohomology class $[F_h]$ is independent of the choice of metric h and depends only on the holomorphic line bundle L.

Definition 1.4. We write $c_1(L) = \frac{i}{2\pi}[F_h] \in H^2(X, \mathbb{R})$ where h is any Hermitian metric in L. This is called the *first Chern class of L*.

What is not apparent from our brief discussion is that

- The class $c_1(L) \in H^2(X, \mathbb{R})$ is actually the image of a class in $H^2(X, \mathbb{Z})$. This lift is what is more normally known as the first Chern class of L. (Notice that the de Rham class will vanish if the integral class is torsion, so the integral class carries strictly more information.)

- In fact, one can use the same definition for *any* unitary connection in L with respect to any Hermitian metric, not just one compatible with the holomorphic structure.

- It follows that the first Chern class depends only on the topological isomorphism class of $L \to X$ (and not its holomorphic structure). These classes can be defined for line bundles over any sufficiently nice topological space (e.g., CW complexes)

- We also remark that one can define higher Chern classes for holomorphic vector bundles of higher rank vector bundles in a similar fashion by constructing differential forms out of their curvature tensors. Again, this gives an image in de Rham cohomology of the genuine topological invariants which live in integral cohomology.

We will not pursue these matters here.

We have seen that when $L \to X$ is a holomorphic Hermitian line bundle, its curvature gives a real $(1,1)$-form $\frac{i}{2\pi}F$ representing $c_1(L)$.

Question 1.5. Given a $(1,1)$-form $\Phi \in -2\pi i c_1(L)$ is there a Hermitian metric h in L with $F_h = \Phi$?

Fix a reference metric h_0. Then $h = e^f h_0$ is the metric we seek if and only if f solves

$$\bar{\partial}\partial f = \Phi - F_{h_0}.$$

This question is the basic prototype of more difficult questions which we will encounter later.

Of course, for this discussion to be of interest, one must have some holomorphic line bundles in the first place. There is always one holomorphic line bundle you are guaranteed to have to hand:

Definition 1.6. Let X be a complex manifold. The holomorphic line bundle $K = \Lambda^n(T^*X)$ is called the *canonical line bundle* and its dual K^* the *anti-canonical line bundle*. The *first Chern class of* X is defined by $c_1(X) = c_1(K^*) = -c_1(K)$.

Exercises 1.1.

1. Prove Proposition 1.2 by following the same proof as was given above for line bundles.

2. Let $L \to \mathbb{C}^n$ be the trivial bundle with Hermitian metric $h = e^{-|z|^2}$. Compute the curvature of the corresponding Chern connection.

3. Let $L \to \mathbb{C}$ be the trivial bundle with Hermitian metric $h = 1 + |z|^2$. Compute the curvature F of the corresponding Chern connection. Calculate $\int_{\mathbb{C}} F$.
4. Given line bundles L_1, L_2, prove that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.
5. Given a vector bundle E, we can define $c_1(E) = c_1(\det E)$ where $\det E$ is the top exterior power of E.
   (a) Prove for vector bundles E_1, E_2 that $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$.
   (b) Prove that if L is a line bundle and E a vector bundle of rank r then $c_1(L \otimes E) = rc_1(L) + c_1(E)$.

1.2 Definitions and examples of Kähler manifolds

Let X be a complex manifold and write $J\colon TX \to TX$ for the endomorphism of the tangent bundle given by multiplication by i.

Definition 1.7. A Riemannian metric g on X is called *Hermitian* if $g(Ju, Jv) = g(u, v)$ for all $u, v \in TX$.

Note that this is equivalent to saying that the bilinear form $\omega(u, v) = g(Ju, v)$ is skew and of type $(1, 1)$. The fact that g is positive definite implies that ω is positive on all complex lines.

Definition 1.8. A real $(1, 1)$-form is called *positive* if it is positive on all complex lines.

Notice that g can be recovered from ω and J via $g(u, v) = \omega(u, Jv)$. This means that specifying a Hermitian metric g on X is the same thing as specifying a positive $(1, 1)$-form ω.

Definition 1.9. Given a Hermitian metric g, we call ω the *associated* $(1, 1)$-*form* of g.

A Kähler manifold is a complex manifold with a Hermitian metric which also satisfies a differential compatibility condition.

Proposition 1.10. *Let (X, J, g) be a Hermitian manifold. The following are equivalent:*

1. *The complex structure J is parallel with respect to the Levi-Civita connection, i.e., $\nabla J = 0$.*

2. *The Chern connection and Levi-Civita connection on TX are the same.*

3. *The associated $(1,1)$-form ω is parallel: $\nabla\omega = 0$.*

4. *The associated $(1,1)$-form ω is closed: $\mathrm{d}\omega = 0$.*

5. *Locally, one can write $\omega = i\bar{\partial}\partial\phi$ for a real valued function ϕ, called a* local Kähler potential.

6. *There exist holomorphic coordinates $z_1, \dots, z_n$ in which the metric is Euclidean to second order: $g = \sum \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_i + O(|z|^2)$.*

Definition 1.11. When one, and hence all, of the above conditions are met we call (X, J, g) a *Kähler* manifold.

Examples 1.12.

1. Let (X, J) be a Riemann surface and let g be a Hermitian metric with associated $(1,1)$-form ω. Since there are no 3-forms on a surface, $\mathrm{d}\omega = 0$ and (X, J, g) is Kähler.

2. Let (X, g) be an oriented surface (real dim 2) with a Riemannian metric. Define $J \colon TX \to TX$ as a positive rotation by $\pi/2$. Isothermal coordinates for X are coordinates in which the metric has the form $g = f(x,y)(\mathrm{d}x^2 + \mathrm{d}y^2)$. It is an important fact that such coordinates always exist. Notice that the transition maps between isothermal coordinate charts are exactly those which are holomorphic with respect to the variable $z = x + iy$. This tells us that J is in fact induced by a holomorphic atlas on X. So J is a genuine complex structure, g is Hermitian with respect to J and so (X, J, g) is Kähler as above.

3. Let (X, J, ω) be Kähler and $Y \subset X$ a complex submanifold. The restriction of the Kähler metric to Y has associated $(1,1)$-form given by the restriction of ω. Since ω is closed, so too is its restriction. Hence the induced metric on Y his again Kähler.

4. The Fubini–Study metric on $\mathbb{CP}^n$ is Kähler. There are various ways to see this. One can either compute in a local unitary chart, to see that $\mathrm{d}\omega = 0$, or use symmetry arguments to see that $\nabla J = 0$.

5. The previous two observations combine to give a plethora of examples: *any complex submanifold of $\mathbb{CP}^n$ inherits a Kähler metric*. To find many such submanifolds, one can look at sets locally cut out as the common zeros of homogeneous polynomials in n variables.

Exercises 1.2.

1. Prove the equivalence of the various definitions of Kähler by proving the chain of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$ in Proposition 1.10. Hint: to prove $4 \Rightarrow 5$ you might like to use the Poincaré lemma which states that if α is a d-closed p-form then locally one can write $\alpha = \mathrm{d}\beta$ for a $(p-1)$-form, together with analogous results for ∂ and $\bar{\partial}$.

2. Consider the hyperbolic metric on the unit disc $D = \{|z| < 1\}$ given by
$$g = \frac{\mathrm{d}x^{+}\mathrm{d}y^2}{(1 - x^2 - y^2)^2}$$
Find a global function $\phi\colon D \to \mathbb{R}$ so that the associated $(1,1)$-form of g is given by $\omega = i\bar{\partial}\partial\phi$.

3. Let $U \subset \mathbb{CP}^n$ be an open set and $f\colon U \to \mathbb{C}^{n+1} \setminus 0$ a local section of the projection map. Prove that the $(1,1)$-form $\omega_{U,f} = -i\bar{\partial}\partial \log |f|$ is positive and that in fact it doesn't depend on the choice of section f.

   Deduce that there is a $\mathrm{U}(n+1)$-invariant Kähler metric on $\mathbb{CP}^n$ which agrees with each $\omega_{U,f}$.

   (This is the Fubini–Study metric.)

4. Prove that there is a unique Riemannian metric on $\mathbb{CP}^n$, up to scale, which is invariant with respect to the action of $\mathrm{U}(n+1)$.

5. Prove that for both of the line bundle metrics from Exercises 1.1(2) and (3), the curvatures are of the form $F = -2\pi i\omega$ where ω is a positive $(1,1)$-form.

1.3 The Kähler identities

Just as on a Riemannian manifold one can define the L^2-adjoint d^* of the exterior derivative in terms of the Hodge star $\mathrm{d}^* = \pm * \mathrm{d} *$, one can do similarly for ∂^* and $\bar{\partial}^*$ on a Hermitian manifold. One of the fundamental facts for Kähler manifolds is the interaction of these operators and the map $L\colon \Lambda^p \to \Lambda^{p+2}$ given by taking the wedge-product with the Kähler form ω.

Proposition 1.13 (The Kähler identities)**.** *On a Kähler manifold, the following hold*
$$[\bar{\partial}^*, L] = i\partial, \qquad [\partial^*, L] = -i\bar{\partial}.$$

To prove these identities, note first that they only see first order derivatives of the Kähler structure. This means that by part 6 of Proposition 1.10 that it suffices to prove them for the flat metric on $\mathbb{C}^n$.

On a Riemannian manifold, we can define the Laplacian on forms:

$$\Delta_{\mathrm{d}} = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$$

On a Hermitian manifold, we can do similarly with ∂ and $\bar{\partial}$:

$$\Delta_{\partial} = \partial^*\partial + \partial\partial^*, \qquad \Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$

In general these Laplacians have little to do with each other, but on a *Kähler manifold* it is a corollary of the Kähler identities that they are all essentially one and the same:

Corollary 1.14. *On a Kähler manifold,*

$$\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_{\mathrm{d}}$$

Moreover, denoting this common operator by Δ, we have the formula

$$\Delta f\, \omega^n = n\, i\bar{\partial}\partial f \wedge \omega^{n-1}$$

Or, equivalently, $\Delta f = \langle i\bar{\partial}\partial f, \omega \rangle$.

Notice that our convention is that Δ is one-half of the usual Riemannian Laplacian!

This result has profound implications for the cohomology of compact Kähler manifolds, called the Hodge theorem. We do not, unfortunately, have time to go into the details here.

Exercises 1.3.

1. Prove the Kähler identities on $\mathbb{C}^n$ and hence on any Kähler manifold.

2. Prove the formulae in Corollary 1.14.

3. Let X be a compact Kähler manifold. Let θ be a $(0,1)$-form with $\bar{\partial}\theta = 0$. Prove that there is a function u such that $\bar{\partial}^*\theta = \bar{\partial}^*\bar{\partial}u$ and hence that $\theta - \bar{\partial}u$ is d- and ∂-closed and coclosed.

1.4 The $\bar{\partial}\partial$-lemma and curvature of line bundles

Definition 1.15. Given a Kähler manifold (X, J, ω), the cohomology class $[\omega] \in H^2(X, \mathbb{R})$ is called the *Kähler class*.

Lemma 1.16 (The $\bar{\partial}\partial$-lemma)**.** *Let (X, J, ω) be a compact Kähler manifold and let α_1, α_2 be cohomologous real $(1,1)$-forms. Then there exists $\phi\colon X \to \mathbb{R}$ such that $\alpha_1 = \alpha_2 + i\bar{\partial}\partial\phi$. Such a function ϕ is unique up to the addition of a constant.*

Corollary 1.17.

1. *Given a holomorphic line bundle $L \to X$ and a real $(1,1)$-form $\Phi \in -2\pi i c_1(L)$ there is a unique Hermitian metric h, up to constant scale, with $F_h = \Phi$.*

2. *If ω_1, ω_2 are two Kähler metrics in the same cohomology class then there exists a smooth function ϕ, unique up to addition of a constant, such that $\omega_1 = \omega_2 + i\bar{\partial}\partial\phi$.*

Definition 1.18. Given two cohomologous Kähler metrics ω_1, ω_2 a function ϕ satisfying $\omega_1 = \omega_2 + i\bar{\partial}\partial\phi$ is called the *Kähler potential* of ω_1 relative to ω_2.

If one has locally, $\omega = i\bar{\partial}\partial\phi$, then ϕ is called a *local Kähler potential for ω*.

This is one of the most important reasons why Kähler metrics are more tractable than general Riemannian metrics: the metric is determined by a single scalar function, rather than a matrix valued function.

Given a cohomology class κ which contains a Kähler metric, we write $\mathcal{H}$ for the space of all Kähler metrics in κ. The above discussion shows that fixing a reference $\omega \in \mathcal{H}$ identifies $\mathcal{H}$ with an open set in the space of functions modulo constants.

We write $\kappa > 0$ to mean that κ contains Kähler metrics.

If $\kappa = c_1(L)$ for some holomorphic line bundle, we can instead look at the set $\mathcal{M}$ of Hermitian metrics h in L for which $\frac{i}{2\pi}F_h$ is a Kähler metric. Fixing a reference $h \in \mathcal{M}$ identifies $\mathcal{M}$ with an open set in the space of all functions. Sending a Hermitian metric to its curvature gives a surjection $\mathcal{M} \to \mathcal{H}$ with fibres copies of $\mathbb{R}$, coming from the freedom to choose the scale of h given F_h.

Definition 1.19. If a holomorphic line bundle $L \to X$ has the property that $c_1(L)$ contains Kähler metrics, we call L a *positive* line bundle. This is often written in shorthand as $c_1(L) > 0$.

A metric $h \in \mathcal{M}$ is called a *positively curved metric*.

Example 1.20. There is a tautological line bundle $\mathcal{O}(-1) \to \mathbb{CP}^n$ over projective space, which inherits a natural Hermitian metric from the map $\mathcal{O}(-1) \to \mathbb{C}^{n+1}$. This induces a Hermitian structure on its dual $\mathcal{O}(1)$. One can check that the curvature of this metric gives exactly the Fubini–Study metric on $\mathbb{CP}^n$. Hence $\mathcal{O}(1)$ is a positive line bundle.

Recall that a complex submanifold $X \subset \mathbb{CP}^n$ inherits a Kähler metric by restriction of the Fubini–Study metric. The same reasoning shows that the restriction of $\mathcal{O}(1)$ to X is a positive line bundle.

The converse to this result is a famous theorem due to Kodaira. We will sketch the proof of this later on.

Theorem 1.21 (Kodaira)**.** *Let $L \to X$ be a positive holomorphic line bundle over a compact complex manifold. Then there exists a holomorhpic embedding $f\colon X \to \mathbb{CP}^n$ and an isomorphism $f^*\mathcal{O}(1) \cong L$.*

Exercises 1.4.

1. Prove the $\bar\partial\partial$-lemma as follows.

   Let $\alpha = \mathrm{d}\beta$ be a real $(1,1)$-form. By applying the results of Exercise 1.3(3) to $\theta = \beta^{0,1}$, prove that $\partial\beta = -\bar\partial\partial u$ for some (complex-valued) function u. Deduce that $\alpha = i\bar\partial\partial\phi$ for a real-valued function ϕ.

   Prove moreover that ϕ is unique up to the addition of a constant.

2. Verify the claims of Example 1.20. You might find it helpful to revisit Exercise 1.2(3).

3. Fix a reference $\omega \in \mathcal{H}$ and use Kähler potentials to identify $\mathcal{H}$ with an open set in $C^\infty(X,\mathbb{R})/\mathbb{R}$. Prove that $\mathcal{H}$ is convex with respect to the natural affine structure on the vector space $C^\infty(X,\mathbb{R})/\mathbb{R}$.

1.5 The volume and Ricci curvature of a Kähler manifold

The volume form of a Kähler manifold has a particularly nice description.

Lemma 1.22. *The volume form of a Kähler (or even just Hermitian) metric is $\omega^n/n!$.*

There is an alternative way to think of the volume form of a Kähler manifold which is particularly important. Recall that $K = \Lambda^n(T^*X)$ denotes the canonical bundle of X, a holomorphic line bundle. A Hermitian metric on the anti-canonical bundle K^* is a nowhere vanishing section of $K \otimes \bar{K} = \Lambda^{n,n}$ which is precisely the bundle where volume forms live.

Lemma 1.23. *Given a Hermitian metric on a complex manifold, the induced metric on K^* is given by the volume form $\frac{\omega^n}{n!} \in K \otimes \bar{K}$.*

Given our above obsession with curvatures of line bundles, it is a natural question to wonder what the curvature of K^* is with this Hermitian structure. We will see shortly it is essentially the Ricci curvature of the metric.

First, a few words about the whole curvature tensor of a Kähler metric. Since $\nabla J = 0$, the curvature tensor satisfies certain algebraic constraints. For a general metric, one can think of the curvature tensor as a skew section R of $\Lambda^2 \otimes \Lambda^2$. For a Kähler metric however R is constrained further to lie in $\Lambda^{1,1} \otimes \Lambda^{1,1}$.

This has implications for the Ricci curvature. The Ricci curvature can be thought of as a symmetric bilinear form $\mathrm{Ric} \in S^2(T^*)$. The additional symmetries alluded to above in the Kähler setting, mean that Ric is J-invariant, i.e., $\mathrm{Ric}(Ju, Jv) = \mathrm{Ric}(u, v)$. This means that one can build a $(1,1)$-form ρ from Ric, just as ω is defined via g: $\rho(u, v) = \mathrm{Ric}(Ju, v)$.

Definition 1.24. The form ρ is called the *Ricci form* of the Kähler manifold.

Proposition 1.25. *The curvature of the the anti-canonical bundle (with its induced Hermitian metric) is given by $F = -i\rho$. In particular, the Ricci form is closed and its cohomology class is fixed by J and independent of the Kähler metric: $[\rho] = 2\pi c_1(X)$.*

This fact has an extremely important consequence: *the Ricci curvature of a Kähler metric is determined by its volume form*. More precisely if ω_1 and ω_2 are two Kähler metrics, we can define a function f by

$$e^f = \frac{\omega_1^n}{\omega_2^n}$$

The corresponding metrics on K are related by $h_1 = e^f h_2$. It follows that the Ricci forms differ by $\rho_1 = \rho_2 + i\bar{\partial}\partial f$.

Prescribing the Ricci curvature of a Kähler manifold is the same as prescribing its volume. In particular, this is zeroth order in the metric, and second order in the Kähler potential!

Exercises 1.5.

1. Prove Lemma 1.22.

   Deduce that if $X \subset \mathbb{CP}^n$ is a complex submanifold then its volume is a positive integer.

2. Prove Lemma 1.23.

3. Prove Proposition 1.25.

4. Compute the Ricci form of the Fubini–Study metric on $\mathbb{CP}^n$.
5. Compute the Ricci form of the hyperbolic disk (Exercise 1.2(2)).

2 The Calabi conjecture and Kähler–Einstein metrics

2.1 The Calabi–Yau theorem

In a series of papers (the first dating from the 1950s, the second from the 1980s) Calabi posed questions which have subsequently driven a large part of research in Kähler geometry. The first such question was the following conjecture, since proved by S.-T. Yau (a result which won him the Fields medal).

Theorem 2.1 (The Calabi–Yau theorem)**.** *Let X be a Kähler manifold and κ a Kähler class on X. Given any real $(1,1)$-form ρ representing $2\pi c_1(X)$, there is a unique Kähler metric in κ with Ricci form ρ.*

Equivalently, if V is any volume form with total volume equal to $\langle \kappa^n, [X] \rangle / n!$ then there is a unique Kähler metric $\omega \in \kappa$ with volume form $\omega^n / n! = V$.

Corollary 2.2. *Suppose X is a compact Kähler manifold with $c_1(X) = 0$. Then each Kähler class on X contains a unique Ricci flat Kähler metric.*

A Kähler manifold with $c_1(X) = 0$ is called a *Calabi–Yau* manifold, in reference to this result. (Although be warned there are other, more stringent, versions of the definition of Calabi–Yau manifolds.)

The first step in the proof is to express the problem as a Monge–Ampère equation.

Definition 2.3. Let $\omega \in \kappa$ be a reference metric. Given a Kähler potential ϕ, write $\omega_\phi = \omega + i\bar{\partial}\partial\phi$. The map $M \colon \mathcal{H} \to C^\infty(X, \mathbb{R})$ defined by

$$M(\omega_\phi) = \frac{\omega_\phi^n}{\omega^n}$$

is called the *Monge–Ampère operator*. We will often just write $M(\phi) = M(\omega_\phi)$ when we think of M as acting on functions.

Now define a function f by $V = e^f \omega^n / n!$. We seek ϕ such that $M(\phi) = e^f$. In this language, the Calabi–Yau theorem amounts to the following:

Theorem 2.4 (Yau)**.** *Let (X, ω) be a compact Kähler manifold. Define the Monge–Ampère operator as above (with reference to ω). Then given any f with $\int e^f \omega^n = \int \omega^n$, there is a solution ϕ, unique modulo additive constants, to the equation $M(\phi) = e^f$. Moreover, ω_ϕ is a positive $(1,1)$-form*

We will discuss some (but not all!) of the steps in the proof of this in the exercises in this and the subsequent section.

Exercises 2.1.

1. (a) Suppose that $\phi \in C^2$. Prove that at a maximum of ϕ, $M(\phi) \geq 1$, whilst at a minimum of ϕ, $M(\phi) \leq 1$.

   (Hint one can find holomorphic coordinates at a point p in which the metric is Euclidean at p *and* in which the complex Hessian $i\bar{\partial}\partial\phi$ of ϕ is diagonal at p.)

   (b) Prove that if ϕ is C^2 and $M(\phi) > 0$ then ω_ϕ is a positive $(1,1)$-form. (Hint: show that if ω_ϕ vanishes on a complex line then $M(\phi) = 0$.)

2. Let $\omega \in \mathcal{H}$ and V be a volume form with total volume equal to that of ω. Let $\{\phi_t : t \in [0,1]\}$ denote a path of Kähler potentials with $\phi_0 = 0$, giving a path of Kähler metrics $\omega_t = \omega_{\phi_t}$. Define

$$E = \int_0^1 \left[\int_X \frac{\partial \phi}{\partial t} \left(\frac{\omega_t^n}{n!} - V \right) \right] \mathrm{d}t$$

   (a) Prove that E depends only on ω_1 and not on the path of Kähler potentials joining it to ω, hence it defines a function $E \colon \mathcal{H} \to \mathbb{R}$ by setting $E(\omega_1)$ equal to the above integral for any choice of path ϕ_t.

   (b) Prove that $\omega_1 \in \mathcal{H}$ is a critical point of E if and only if ω_1 has volume form $\omega_1^n/n! = V$.

   (c) Let $\psi \in C^\infty(X, \mathbb{R})$ be a non-constant function and consider the corresponding linear path $\omega_s = \omega_{s\psi}$ in $\mathcal{H}$. Prove that $E(\omega_s)$ is strictly convex in s.

   Deduce that if a solution to the Calabi conjecture exists, it must be unique.

2.2 Kähler–Einstein metrics

In order to admit a Ricci flat Kähler metric, a Kähler manifold X must have $c_1(X) = 0$. The Calabi–Yau theorem tells us this is also sufficient. One can also of course consider other types of Einstein metric.

Definition 2.5. A Riemannian metric is called *Einstein* if $\mathrm{Ric} = \lambda g$ where λ is a constant, called the *Einstein constant*.

Example 2.6. A Kähler metric on a Riemann surface is Einstein if and only if it has constant curvature. Every Riemann surface carries such a metric which is unique except in the case of $\mathbb{CP}^1$, where there is a 3 dimensional family of round metrics.

In higher dimensions, Einstein metrics are difficult to find. But in Kähler geometry they are, as we will see, especially abundant.

First we point out that there is an "obvious" necessary condition. A Kähler metric is Einstein precisely when its Kähler and Ricci forms are proportional: $\rho = \lambda\omega$. Recall that $[\rho] = 2\pi c_1(X)$. So if $\lambda > 0$ it is necessary that $c_1(X) > 0$, i.e., that the anti-canonical bundle be positive, whilst if $\lambda < 0$ it is necessary that $c_1(X) < 0$, i.e., that the canonical bundle be positive.

(Warning, the notation here is misleading: there are certainly times when none of $c_1(X) > 0$, $c_1(X) > 0$ or $c_1(X) = 0$ is true!)

Definition 2.7. A complex manifold with $c_1(X) > 0$—i.e., with positive anti-canonical bundle—is called a *Fano* manifold.

(Such manifolds are rare. Indeed it is know that in each dimension there is a finite number of deformation classes of Fano manifolds. In complex dimension 2, they are the so-called Del Pezzo surfaces, blow-ups of $\mathbb{CP}^2$ at at most 8 points in sufficiently general position. In complex dimension 3 there are 105 different deformation types of Fanos.)

We will see that when $c_1(X) \leq 0$, this necessary condition is also sufficient for the existence of a Kähler–Einstein metric. However, when $c_1(X) > 0$ there are obstructions to existence and the whole question is far more subtle (and as yet currently unresolved).

We saw above how to write a Ricci flat Kähler metric as a Monge–Ampère equation. We will now do the same for non-zero Einstein constants.

By scaling we can reduce to the case $\lambda = \pm 1$. Assume that $\lambda c_1(X) > 0$, which is the necessary condition for existence of a Kähler–Einstein metric. Let ω be a reference metric with $2\pi c_1(X) = \lambda[\omega]$ and write ρ for the Ricci form of ω. Write also ρ_ϕ for the Ricci form of ω_ϕ. We want to solve $\rho_\phi = \lambda\omega_\phi$, an equation for the potential ϕ which we now rewrite in terms of the Monge–Ampère operator M.

Recall that $M \colon \mathcal{H} \to C^\infty(X, \mathbb{R})$ is defined by $M(\omega_\phi) = \omega_\phi^n / \omega^n$. The Ricci forms ρ_ϕ and ρ are then related by

$$\rho_\phi = \rho + i\bar{\partial}\partial \log M(\phi)$$

Since ρ and $\lambda\omega$ are in the same cohomology class we also know, by the $\bar{\partial}\partial$-lemma, that there is a function f such that $\rho = \lambda\omega + i\bar{\partial}\partial f$. (This function

f is often called the *Ricci potential of* ω.) Meanwhile $\omega_\phi = \omega + i\bar{\partial}\partial\phi$. So $\rho_\phi = \lambda\omega_\phi$ becomes

$$i\bar{\partial}\partial(f + \log M(\phi)) = i\lambda\bar{\partial}\partial\phi$$

In other words, we want to find $\phi\colon X \to \mathbb{R}$ such that

$$M(\phi) = e^{f+\lambda\phi}.$$

This is again a Monge–Ampère equation.

As mentioned above, Yau proved the existence of a solution in the case $\lambda = 0$. When $\lambda = -1$, existence was proved independently by Aubin and Yau.

Theorem 2.8 (Aubin, Yau)**.** *Let (X, ω) be a compact Kähler manifold. Given any smooth function $f\colon X \to \mathbb{R}$, the equation $M(\phi) = e^{f-\phi}$ has a unique solution (where M is the Monge–Ampère operator as defined above, with reference to the metric ω).*

It follows that if $c_1(X) < 0$, there is a unique Kähler–Einstein metric on X, up to scale (whose Einstein constant is necessarily negative).

The proof of this is outlined in the exercises. A key step involves a C^0 estimate on a solution of $M(\phi) = e^{f+\lambda\phi}$ in terms of f. This is fairly straightforward when $\lambda < 0$. When $\lambda = 0$ (the case of the Calabi conjecture) the proof is much more involved.

When $\lambda > 0$ the hoped-for bound is known to be false and things are very different. There are obstructions to the existence of Kähler–Einstein metrics on Fano manifolds, some of which we will see later. The full question of deciding when such a metric exists is still an open problem. We will state a famous conjecture of Donaldson, Tian and Yau about this later on.

Warning! *Our conventions differ from those often used in the literature, where you will find, for example, the Monge–Ampère operator defined via the equation $M(\phi) = (\omega - i\bar{\partial}\partial\phi)^n/\omega^n$. Changing from our notation to this just amounts to swapping the sign of ϕ, but this can have the disconcerting effect of seeming to send λ to $-\lambda$ in the Monge–Ampère equation!*

Exercises 2.2. The goal of these exercises is to sketch the proof of Theorem 2.8. The idea is to prove that the set of functions f for which $M(\phi) = e^{f-\phi}$ has a solution is both open and closed. Then, by connectedness, it will be solvable for all f. This is often referred to as the *continuity method*.

1. Write $U \subset C^{5,\alpha}$ for the set of $\phi \in C^{5,\alpha}$ for which ω_ϕ is a positive $(1,1)$-form.

Prove that the map F defined on smooth Kähler potentials given by

$$F(\phi) = \log M(\phi) + \phi$$

extends to a map $U \to C^{3,\alpha}$.

Write $S \subset C^{3,\alpha}$ for the image of F. We will show S is both open and closed.

2. Prove that the derivative of F at ϕ is given by $DF_\phi(\psi) = \Delta_\phi(\psi) + \psi$, where Δ_ϕ is the Laplacian of ω_ϕ.

   Deduce that S is open.

3. Let f_n be a sequence in S which converges to f in $C^{3,\alpha}$ and let $\phi_n \in U$ solve $F(\phi_n) = f_n$. To prove S is closed we will show that a subsequence of the ϕ_n converges to a solution of $F(\phi) = f$. There are several steps.

   (a) Step 1, C^0 bound.
   Prove that if $\phi \in C^2$, then $\|\phi\|_{C^0} \leq \|F(\phi)\|_{C^0}$.
   (Hint: go back to Exercise 2.1(1).)

   (b) Step 2, $C^{2,\alpha}$ bound given the C^0 bound.
   For this you can quote the following result (or if you're brave try and prove it yourself!)

   Proposition. *Let W be a set of C^5 Kähler potentials, which are uniformly bounded in C^0. If the set $\{F(\phi) : \phi \in W\}$ is bounded in C^3 then W is bounded in $C^{2,\alpha}$ for any $0 < \alpha < 1$.*

   (This part also holds for $\lambda \geq 0$)

   (c) Step 3, regularity.
   Prove that if $\phi \in C^2$ and $F(\phi) \in C^{r,\alpha}$ then $\phi \in C^{r+2,\alpha}$.
   (This part also holds for $\lambda \geq 0$)

   (d) Deduce that S is closed and hence complete the proof of Theorem 2.8.

3 Extremal Kähler metrics

3.1 Calabi energy

There is an old question (going back at least as far as Berger?) to find a "best Riemannian metric" on a given manifold. In the Kähler setting this vague question can be made extremely precise.

If one supposes that $c_1(X)$ is either zero or definite then Kähler–Einstein metrics provide ideal candidates for "best metrics" on the manifold. Calabi's next contribution was to define a notion of "best" which works for any Kähler class.

Calabi's idea is to try and minimise the function $C\colon \mathcal{H} \to \mathbb{R}$ which is defined by

$$C(\omega) = \int_X S(\omega)^2 \frac{\omega^n}{n!}$$

where $S(\omega)$ is the scalar curvature of ω.

Lemma 3.1. *For Kähler metrics in a fixed cohomology class, the following quantities differ by topological constants, i.e., constants depending only on X and $[\omega]$:*

$$\int_X S(\omega)^2 \frac{\omega^n}{n!}, \qquad \int_X |\operatorname{Ric}(\omega)|^2 \frac{\omega^n}{n!}, \qquad \int_X |R(\omega)|^2 \frac{\omega^n}{n!}.$$

(Here the pointwise norms of tensors are taken with respect to the metric ω; R is the full curvature tensor of ω.)

Because of this, minimising C amounts to minimising the L^2-norm of curvature over $\mathcal{H}$. So a minimum (if it exists!) can be thought of as the "least curved" metric in a given cohomology class.

Definition 3.2. The quantity $C(\omega)$ is called the Calabi energy of ω.

To compute the Euler–Lagrange equations of Calabi energy, one needs the following formulae for the variation of scalar curvature.

Lemma 3.3. *Given $\phi \in C^\infty(X, \mathbb{R})$ and $\omega \in \mathcal{H}$, write $\omega_t = \omega + ti\bar{\partial}\partial\phi$. Then, at $t = 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(\omega_t) = \Delta^2\phi - \langle \rho, i\bar{\partial}\partial\phi \rangle$$

where all geometric quantities are computed with respect to ω.

It turns out that infinitesimal changes in scalar curvature are intimately related to deformations of the data (X, J, ω) to explain this relation, we need some notation.

Definition 3.4. Let $\mathcal{D}\colon C^\infty(X, \mathbb{R}) \to \Omega^{0,1}(TX)$ be the operator defined by

$$\mathcal{D}(f) = \bar{\partial}(\xi_f)$$

where ξ_f is the Hamiltonian vector field corresponding to f. I.e., for any other vector field v, $\omega(\xi_f, v) = v \cdot f$.

So $\mathcal{D}(f)$ measures the failure of the Hamiltonian flow of f to be holomorphic. Since the Hamiltonian flow of f automatically preserves ω, and the flow of a holomorphic vector preserves J, when $f \in \ker \mathcal{D}$, ξ_f is and infinitesimal automorphisms of (X, J, ω). In fact, when $b_1(X) = 0$, *all* symplectic vector fields on X are Hamiltonian and so $\ker \mathcal{D}$ is exactly the infinitesimal automorphisms of (X, J, ω) plus constants.

Lemma 3.5.
$$\mathcal{D}^*\mathcal{D}(\phi) = \Delta^2\phi - \langle \rho, i\bar{\partial}\partial\phi \rangle + \frac{1}{2}\langle \nabla S, \nabla \phi \rangle$$
where $\mathcal{D}^*$ is the L^2 adjoint of $\mathcal{D}$.

Proposition 3.6. *Given $\phi \in C^\infty(X, \mathbb{R})$ and $\omega \in \mathcal{H}$, write $\omega_t = \omega + it\bar{\partial}\partial\phi$. Then, at $t = 0$,*
$$\frac{\mathrm{d}}{\mathrm{d}t} C(\omega_t) = \int_X \phi \, \mathcal{D}^*\mathcal{D}S(\omega) \, \frac{\omega^n}{n!}$$
Hence ω is a critical point of $C \colon \mathcal{H} \to \mathbb{R}$ if and only if the Hamiltonian flow of $S(\omega)$ is holomorphic.

Definition 3.7. A Kähler metric for which $\mathcal{D}S = 0$ is called *extremal*.

Note that if X admits no non-zero holomorphic vector fields, then an extremal metric automatically has constant scalar curvature.

Lemma 3.8. *Let ω be a Kähler metric of constant scalar curvature and suppose that $\lambda[\omega] = 2\pi c_1(X)$ for some λ. Then ω is in fact Kähler–Einstein: $\rho = \lambda\omega$.*

So constant scalar curvature metrics are a generalisation of Kähler–Einstein metrics which can be looked for in any Kähler class.

Lemma 3.9. *The mean value of the scalar curvature of $\omega \in \mathcal{H}$ does not depend on the choice of ω, only on X and $[\omega]$.*

Proof. $\int_X S\,\omega^n = \int n\rho \wedge \omega^{n-1} = 2\pi n \langle c_1(X) \cdot [\omega]^{n-1}, [X] \rangle$ which is independent of the choice of metric in the class $[\omega]$. ∎

This means that when looking for a constant scalar curvature metric one at least knows what constant to aim for!

Exercises 3.1.

1. Given a real $(1,1)$-form ρ, derive a formula for $|\rho|^2\omega^n$ in terms of $\rho \wedge \rho \wedge \omega^{n-2}$ and $(\Lambda\rho)^2\omega^n$.

   Deduce that $\int_X |\operatorname{Ric}|^2\omega^n$ and $\int_X S\omega^n$ differ by a constant which depends only on X and the Kähler class $[\omega]$ but not on the metric ω itself.

2. By differentiating the formula $S\omega^n = n\rho \wedge \omega^{n-1}$, prove Lemma 3.3.

   Can you prove Lemma 3.5?

3. Prove Proposition 3.6.

4. Using the Kähler identities, prove that a metric has constant scalar curvature if and only if its Ricci form is harmonic.

   Deduce Lemma 3.8

3.2 Some examples of extremal metrics

Calabi's first examples. In the paper introducing extremal metrics, Calabi also provided the first non-trivial examples (i.e., with non-constant scalar curvature). He considered metrics on the projective completion X_k of $\mathcal{O}(k) \to \mathbb{CP}^{n-1}$ which are invariant under the action of $\mathrm{U}(n)$. The generic orbits of this action have codimension 1 and so the partial differential equation $\bar{\partial}\nabla S = 0$ becomes an ordinary differential equation which one can solve.

More explicitly, the complement of the zero and infinity sections of $M_k \to \mathbb{CP}^{n-1}$ is covered by a single chart with image $\mathbb{C}^n \setminus 0$, in which the $\mathrm{U}(n)$-action is standard. One then considers Kähler potentials which depend only on the $\mathrm{U}(n)$-invariant variable $t = \log \sum |z_j|^2$. So one puts

$$\phi(z, \bar{z}) = u(t)$$

where $u \colon \mathbb{R} \to \mathbb{R}$ must satisfy certain conditions as $t \to \pm\infty$ to correspond to a Kähler potential of a metric on $\mathbb{C}^n \setminus 0$ which extends to the whole of X_k.

One then converts the extremal metric equation into an ODE for u which can then be shown to have a solution with the required boundary conditions.

The theorems of Hong and Brönnle. The next theorems we mention also concern ruled manifolds, i.e., of the form $\mathbb{P}(E)$ where $E \to Y$ is a holomorhpic vector bundle. To state these we will need the definition of a Hermitian–Einstein connection.

Definition 3.10. Given a holomorphic vector bundle $E \to Y$ over a Kähler manifold (Y, θ), a Hermitian metric in E is called *Hermitian–Einstein* if the curvature $F \in \Omega^{1,1}(\mathfrak{u}(E))$ of the Chern connection satisfies the equation

$$\langle F, \theta \rangle = c \cdot \mathrm{Id}$$

for a constant c.

The constant here is topological (just as for the mean value $\bar{S}$ of the scalar curvature). It is determined by the *slope* of E:

$$\mu(E) = \frac{\langle c_1(E) \wedge \theta^{n-1}, [X] \rangle}{\operatorname{rank} E}$$

If E admits a Hermitian–Einstein connection then

$$c = \frac{2\pi\mu(E)}{(n-1)!V}$$

where V is the volume of Y.

The theorems of Hong and Brönnle concern so-called *adiabatic* Kähler classes on $\mathbb{P}(E)$. First, note that the fibrewise tautological bundles fit together to give a line bundle over $\mathbb{P}(E)$. We denote the dual of this bundle by $L \to \mathbb{P}(E)$. Note that on each fibre, L is the hyperplane bundle of that projective space. The classes that Hong and Brönnle consider are of the form $\kappa_r = c_1(L) + r\pi^*\kappa$ for r large, where $\pi \colon \mathbb{P}(E) \to Y$ is the projection and κ is a Kähler class on the base.

Theorem 3.11 (Hong)**.** *Let $E \to Y$ be a simple holomorphic vector bundle over a Kähler manifold. Assume that the class κ admits a constant scalar curvature metric θ and that E admits a Hermitian–Einstein metric with respect to this θ. Finally assume that Y has no holomorphic vector fields. Then for all large r, the class κ_r on $\mathbb{P}(E)$ admits a constant scalar curvature metric.*

Theorem 3.12 (Brönnle)**.** *Let (Y, θ) be a compact Kähler manifold with constant scalar curvature and no holomorphic vector fields. Let $V \to Y$ be a holomorphic vector bundle which splits as a direct sum $V = E_1 \oplus \cdots \oplus E_r$, where each E_j is as in Hong's theorem. Suppose moreover that all of the E_j have different slopes. Then for all large r, the class κ_r on $\mathbb{P}(V)$ admits an extremal Kähler metric.*

3.3 Futaki's invariant

We next discuss an obstruction to the existence of constant scalar curvature Kähler metrics (and in particular Kähler–Einstein metrics) introduced by Futaki. Given a Kähler class κ, the Futaki invariant associates to each holomorphic vector field v on X a complex number $F(v)$.

Throughout this section we use "holomorphic vector field" to mean a section v of TX for which $L_v J = 0$. Given such a vector field, $v^{1,0}$ is then a holomorphic section of $TX^{1,0}$ in the usual sense. Conversely, given a holomorphic section of $TX^{1,0}$, its real part v has the property that $L_v J = 0$.

To begin with, we will assume that $\kappa = c_1(L)$ and that the vector field v lifts to a vector field $\hat{v}$ on L which preserves the fibrewise linear structure.

To define $F(v)$ we will also pick a Hermitian metric h in L whose curvature $F = -2\pi i\omega$ defines a Kähler metric on X.

We can split $\hat{v}$ into vertical and horizontal pieces using the Chern connection A in L:

$$\hat{v} = v^\flat + f\xi$$

where $v^\flat$ is the horizontal lift of v via A, ξ is the generator of the S^1-action on L and $f\colon X \to \mathbb{C}$ is a complex valued function giving the vertical component of $\hat{v}$. Note that f is determined up to an overall constant by the fact that $\bar{\partial} f = (\iota_v \omega)^{0,1}$, which follows from the fact that $\hat{v}$ is holomorphic. We now define

$$F(h, v) = \int_X (S - \bar{S}) f \, \frac{\omega^n}{n!}$$

where

$$\bar{S} = \frac{1}{V} \int_X S(\omega) \frac{\omega^n}{n!}$$

is the average value of the scalar curvature of Kähler metrics in $\mathcal{H}$.

It may seem at first sight that this quantity depends on our choice of Hermitian metric h in L, but one can show by differentiating the formula with respect to h that this is not actually the case.

There is an alternative formula for F involving the Greens operator G (the inverse of the Laplacian on functions). Set $g = G(S - \bar{S})$, then one can check that

$$F(\omega, v) = \int_X v^{1,0} \cdot g \, \frac{\omega^n}{n!}$$

agrees with the previous definition of F. The second version has the advantage that it makes sense for arbitrary Kähler classes and holomorphic vector fields. To define g one needs to select $\omega \in \mathcal{H}$, as the notation indicates, but again the dependence on ω is illusory.

Theorem 3.13 (Futaki)**.** *The quantity $F(\omega, v)$ above does not depend on the choice of $\omega \in \mathcal{H}$.*

Write $\mathfrak{h}(X)$ for the space of all holomorphic vector fields on X.

Definition 3.14. The map $F\colon \mathfrak{h}(X) \to \mathbb{C}$ defined by $F(v) = F(\omega, v)$ for some $\omega \in \mathcal{H}$ is called the *Futaki invariant* of $\mathcal{H}$.

The following is immediate.

Lemma 3.15. *If there is a constant scalar curvature metric in $\mathcal{H}$, then $F = 0$.*

The proofs of the next two Lemmas are exercises.

Lemma 3.16. *If $F = 0$ then any extremal metric in $\mathcal{H}$ actually has constant scalar curvature.*

Lemma 3.17. *If $u, v \in \mathfrak{h}(X)$ then $F([u, v]) = 0$. In other words, $F \colon \mathfrak{h}(X) \to \mathbb{C}$ is a character.*

Exercises 3.2.

1. Prove Theorem 3.13 in the case that v lifts to a holomorphic vector field $\hat{v}$ on L. To do this, let h_0 be a positively curved metric in L and consider the path $h_t = e^{2\pi t\phi} h_0$, where $\phi \in C^\infty(X, \mathbb{R})$. Now prove that the derivative of $F(h_t, v)$ with respect to t is zero.

2. Prove Lemma 3.16 by considering the Futaki invariant of the holomorphic vector field $\nabla S = J\xi_S$.

3. The aim of this question is to prove Lemma 3.17. Let $u, v \in \mathfrak{h}(X)$ be holomorphic vector fields and let ω be a Kähler metric.

   Let $f_t \colon L \to L$ be the one-parameter group of biholomorphisms generated by u. Show firstly that $f_t^*\omega$ is Kähler for all t.

   Next, prove that
$$\frac{\mathrm{d}}{\mathrm{d}t} F(f_t^*\omega, v) = F(\omega, [u, v]).$$

3.4 A localisation formula for $F(v)$

The Futaki invariant can be quite awkward to calculate directly. We now state (but give no proof of) a way to compute it as a sum of local contributions from the fixed loci of v.

Definition 3.18. A holomorphic vector field v on a complex manifold X is called *non-degenerate* if the zero set of v is a disjoint union of connected complex submanifolds $\{Z_j\}$ of X. Moreover, we require that at each $z \in Z_j$, the linear map
$$Dv \colon T_z X \to T_z X$$
descends to an isomorphism $T_z X / T_z Z_j$.

In the presence of a Kähler metric, we can identify the quotient $Q_z = T_z X / T_z Z_j$ with the normal $N_z = (T_z Z_j)^\perp$ and then the map induced by Dv is the projection $(\nabla v)^\perp$ of ∇v to N.

In the simplest case, where v has an isolated zero, we can write in coordinates $v = \sum v_j \frac{\partial}{\partial z_j}$ where $v_j(0) = 0$. This zero is then non-degenerate precisely when the following matrix is invertible:

$$\left(\frac{\partial v_i}{\partial z_j}\right)_{i,j=1,\dots,n}$$

On a component of the zero locus, Dv descends to a an isomorphism L_j of the bundle $Q = TX/TZ_j$. Since L_j is holomorphic, its trace $t_j = \mathrm{Tr}(L_j)$ is constant.

We will also need another constant associated to each component Z_j. For this we suppose as before that v lifts to a vector field $\hat{v}$ on the positive line bundle $L \to X$. Choosing a positively curved metric in L we obtain a splitting

$$\hat{v} = v^\flat + f\xi$$

for a complex valued function f which is uniquely determined by v up to the addition of a constant (corresponding to the different lifts of v to L). We know that $\bar{\partial} f = (\iota_v \omega)^{0,1}$ and so f restricts to a holomorphic function on each component Z_j of the zero locus, which implies in fact that f is constant on each Z_j. We write f_j for the value of f on Z_j.

Finally, we note that the normal bundle $N_j = (TZ_j)^\perp$ and the quotient bundle $Q_j = TX/TZ_j$ are canonically isomorphic and so the holomorphic bundle Q_j inherits a Hermitian structure. We write $F_j \in \Omega^{1,1}(\mathfrak{u}(Q_j))$ for the curvature form of this metric.

With the definitions of L_j, t_j, f_j and F_j in hand, we can now state the localisation formula.

Theorem 3.19. *If v is a non-degenerate holomorphic vector field with zero locus $\{Z_j\}$, then*

$$F(v) = \sum_j \int_{Z_j} \frac{\left(t_j + c_1(X)\right)\left(\pi f_j + [\omega]\right)^n - \frac{n\bar{S}}{(n+1)\pi}\left(\pi f_j + [\omega]\right)^{n+1}}{\det\left(L_j + \frac{i}{2\pi}F_j\right)}$$

A word or two is in order about how to interpret this expression. The numerator and denominator of the integrand can be expanded as series whose coefficients are differential forms, so the integrand as a whole is expressible as a series whose coefficients are differential forms. To compute the integral over Z_j we simply keep the part which is of degree equal to the dimension of Z_j. Whilst this is somewhat cumbersome to explain in words, it is straightforward to carry out in practice.

Exercises 3.3.

1. Suppose that $[\omega] = 2\pi c_1(X)$ and that the positive line bundle we are considering is K^*, the anti-canonical bundle.

   (a) Prove that there is a natural lift of any holomorphic vector field v to a field $\hat{v}$ on K^* preserving the fibrewise linear structure.

   (b) Prove that in this case if v is non-degenerate then on each component of its fixed locus, $f_j = t_j$.
   Deduce that for the case of the anti-canonical bundle,

$$F(v) = \frac{\pi^n}{n+1} \sum_j \int_{Z_j} \frac{\left(t_j + c_1(X)\right)^{n+1}}{\det\left(L_j + \frac{i}{2\pi} F_j\right)}$$

2. Prove that if all of the zeros of v are isolated points $z_1, \ldots, z_k$, then

$$F(v) = \pi^n \sum_j \frac{\left(t_j - \frac{n\bar{S}}{n+1} f_j\right) f_j^n}{\det L_j}$$

3. Let X be a complex surface, $[\omega] = 2\pi c_1(X)$. Let v be a non-degenerate holomorphic vector field and write the zero locus of v as a collection of points $\{z_j : j \in J\}$ and curves $\{Z_k : k \in K\}$. Prove that

$$F(v) = \frac{\pi^3}{3} \sum_{j \in J} \frac{t_j^3}{\det L_j} + \frac{\pi^3}{3} \sum_{k \in K} L_k\big(2\langle c_1(X), [Z_k]\rangle + 2 - 2g(Z_k)\big)$$

   (Here, $g(Z_k)$ is the genus of Z_k and we note that on Z_k, L_k is a holomorphic isomorphism of a rank 1 bundle, hence multiplication by a constant, which we also denote by L_k.)

4. Let X denote the blow-up of $\mathbb{CP}^2$ in the point $[1, 0, 0]$.

   (a) Show that the $\mathbb{C}^*$-action on $\mathbb{CP}^2$ induced by the action $(x, y, z) \mapsto (x, \lambda y, \lambda z)$ lifts to X.

   (b) Copmute the Futaki invariant of the generator of this action with respect to the anti-canonical bundle and deduce that X does not admit a Kähler–Einstein metric.

5. Let X denote the blow-up of $\mathbb{CP}^2$ in the points $[1, 0, 0]$ and $[0, 1, 0]$.

   (a) Show that the $\mathbb{C}^*$-action on $\mathbb{CP}^2$ induced by the action $(x, y, z) \mapsto (x, \lambda y, \lambda z)$ lifts to X.

   (b) Copmute the Futaki invariant of the generator of this action with respect to the anti-canonical bundle and deduce that X does not admit a Kähler–Einstein metric.

3.5 An algebro-geometric formula for $F(v)$

There is another way to compute Futaki invariants using a result from algebraic geometry called the Hirzebruch–Riemann–Roch formula. In this instance it is essential that we assume the holomorphic vector field v on X lifts to a holomorphic vector field $\hat{v} = v^\flat + f\xi$ on L where it generates a $\mathbb{C}^*$-action. We now consider the vector spaces $V_k = H^0(X, L^k)$ for all values of k. The first application of Hirzebruch–Riemman–Roch that we need is a formula for the dimension of V_k.

Proposition 3.20. *For all large values of k, the dimension d_k of V_k is given by a polynomial $q(k)$ in k. Explicitly,*

$$q(k) = Ck^n + Dk^{n-1} + \cdots$$

where $n = \dim X$, $C = \int_X \frac{\omega^n}{n!}$ *and* $D = \int_X \frac{\rho \wedge \omega^{n-1}}{(n-1)!}$.

The next quantity we will apply Hirzerbuch–Riemann–Roch to is the weight of the $\mathbb{C}^*$-action on V_k. Since $\mathbb{C}^*$ acts on L it also acts on sections of L^k and hence on V_k and so on the complex line $\Lambda^{d_k} V_k$. Any action of $\mathbb{C}^*$ on a complex line is determined by an integer w, called the weight with $\lambda \in \mathbb{C}^*$ acting as multiplication by λ^w. In our case we obtain for each $\Lambda^{d_k} V_k$ a weight w_k.

Proposition 3.21. *For all large values of k, the weight w_k of the action of $\mathbb{C}^*$ on $\lambda^{d_k} V_k$ is given by a polynomial $p(k)$. Explicitly,*

$$p(k) = Ak^{n+1} + Bk^n + \cdots$$

where $n = \dim X$, $A = \int_X f \frac{\omega^n}{n!}$ *and* $B = \int_X f S(\omega) \frac{\omega^n}{n!}$.

Corollary 3.22. *For large k there is an expansion*

$$\frac{w_k}{kd_k} = \frac{A}{C} - \frac{F(v)}{C} k^{-1} + \cdots$$

where $F(v)$ is the Futaki invariant.

The fact that the Futaki invariant can be read off as the coefficient of k^{-1} in this expansion has two consequences. Firstly, it is often possible to compute w_k and d_k directly, without recourse to the Hirzebruch–Riemann–Roch formulae; this then gives an alternative way to compute $F(v)$. Secondly, and perhaps more importantly, this formulation makes sense for $\mathbb{C}^*$-actions on singular manifolds with positive line bundles. This will be of paramount importance in what follows.

4 The Yau–Tian–Donaldson conjecture with a broad brush

4.1 The Riemannian geometry of $\mathcal{M}$

Recall that $\mathcal{M}$ denotes the space of positive Hermitian metrics in a fixed holomorphic line bundle. Fixing a reference metric h_0 any other metric is of the form $h = e^{2\pi\phi}h_0$ for some function ϕ, which satisfies the inequality that $\frac{i}{2\pi}F_{h_0} + i\bar{\partial}\partial\phi > 0$. Thus we can identify $\mathcal{M}$ with an open set in an affine space modelled on $C^\infty(X, \mathbb{R})$. This affine structure is well adapted to the Calabi conjecture as we saw in Exercise 2.1(2). However, for the study of constant scalar curvature or more generally extremal Kähler metrics, there is another geometry in $\mathcal{M}$ which is better suited. (Almost everything we say in this section applies to the more general case of Kähler metric in an arbitrary Kähler class, where $\mathcal{M}$ should be taken to mean the space of Kähler potentials with respect to some reference metric.)

There is a natural Riemannian metric on $\mathcal{M}$, which was discovered independently by Donaldson, Mabuchi and Semmes, which has some remarkable properties. To define the metric, note that there is a natural identification $T_h\mathcal{M} \cong C^\infty(X, \mathbb{R})$

$$\langle \phi, \psi \rangle_h = \int_X \phi\psi \, \frac{\omega_h^n}{n!}$$

where $\omega_h = \frac{i}{2\pi}F_h$ is the Kähler form associated to h. This innerproduct depends on h and so gives a curved metric on $\mathcal{M}$, not directly compatible with the affine structure.

To describe the Levi-Civita connection, we take a path $h_t = e^{2\pi\phi_t}h_0$ in $\mathcal{M}$ and a path of tangent vectors along h_t, which amounts to a function ψ on $X \times [0, 1]$. A connection on $T\mathcal{M}$ is determined by the derivative $D_t\psi$ of ψ along h_t.

Lemma 4.1. *In the above set-up, the covariant derivative of ψ along h_t is*

$$D_t\psi = \frac{\partial\psi}{\partial t} + \frac{1}{2}\left(\nabla\psi, \nabla\frac{\partial\phi}{\partial t}\right)_{\omega_{h_t}}$$

where the innerproduct on the right hand side is pointwise between vector fields on X, using the metric ω_{h_t} defined by h_t.

To verify this, one must simply check that the connection is both metric and torsion free. (In infinite dimensions the Levi-Civita connection is not guaranteed to exist, but when it does it is unique.) With this definition in hand, the follow facts are the result of calculations.

Proposition 4.2.

1. *The curvature tensor of R is given by*

$$R(\phi,\psi)(\chi) = -\frac{1}{4}\{\{\phi,\psi\}_h,\chi\}_h$$

where $\{\cdot,\cdot\}_h$ is the Poisson bracket of ω_h.

2. *The curvature tensor R is covariant constant: $\nabla R = 0$.*

3. *The sectional curvatures of $\mathcal{M}$ are non-positive. More precisely, at $h \in \mathcal{M}$,*

$$R(\phi,\psi,\phi,\psi) = -\frac{1}{4}\|\{\phi,\psi\}_h\|_h^2$$

where $\|\cdot\|$ is the L^2-norm on functions associated to ω_h.

What is remarkable is that these are identical to the formulae for the curvature of certain symmetric spaces. Let K be a compact Lie group and$G$ its complexification. A choice of bi-invariant Riemannian metric on K makes it a positively curved symmetric space, but one can also construct from here the so-called negatively curved dual. The bi-invariant form on $\mathfrak{k}$ endows G/K with a Riemannian metric which is invariant under the action of G by left multiplication. Given $x \in i\mathfrak{k}$, we write x also for the induced vector field on G/K. Then the curvature tensor of G/K is given by $R(x,y)(z) = -[[x,y],z]$.

Because of this, heuristically at least $\mathcal{M}$ can be thought of as the negatively curved symmetric space dual to the group whose Lie algebra is $C^\infty(X,\mathbb{R})$ endowed with the Poisson bracket of some symplectic form ω. When $\omega = \frac{i}{2\pi}F_A$ for some unitary connection A in a line bundle L, there is just such a group, namely the group of maps $L \to L$, taking fibres isometrically to fibres and which also preserve A. This group should play the rôle of K in the above story. At this point, however, the analogy breaks down: there is no complexification of K.

Despite this, traces of the "phantom group" are still to be found. For example, 1-parameter subgroups $\mathbb{C}^* \subset G$ descend to G/K to give geodesics, so one can think of the geodesics of $\mathcal{M}$ in this way. The following lemma gives describes the geodesic equation.

Lemma 4.3. *A function $\phi\colon X \times \mathbb{R} \to \mathbb{R}$ corresponds to a geodesic $t \mapsto e^{2\pi\phi_t}h$ in $\mathcal{M}$ if and only if*

$$\ddot{\phi} + \frac{1}{2}|\nabla\dot{\phi}|_{h_t}^2 = 0.$$

Note that a general geodesic involves solving a PDE and so existence is not guaranteed as it is in the finite dimensional case (where geodesics are solutions of ODEs). There is one situation in which some geodesics are easy to describe. Suppose that the holomorphic isometry group K of (X, ω, J) has positive dimension. The complexification G of K acts on X preserving J, but not necessarily ω. This defines a map $G/K \to \mathcal{H}$, by pull-back. It is now an exercise to check that geodesics in G/K map to geodesics in $\mathcal{H}$.

It is possible to express the geodesic equation as a degenerate Monge–Ampère equation. Given a function $\phi\colon X \times \mathbb{R} \to \mathbb{R}$ we extend it to a rotationally invariant function $\Phi\colon X \times \mathbb{C}^*$ by $\Phi(x, te^{i\theta}) = \phi(x, t)$. Write Ω_0 for the pull-back of a Kähler metric ω_0 on X to the product $X \times \mathbb{C}^*$ and write $\Omega = \Omega_0 + i\bar{\partial}\partial\Phi$. The following is a calculation.

Lemma 4.4. *d The function ϕ a geodesic in $\mathcal{M}$ if and only if the form Ω satisfies the degenerate Monge–Ampère equation $\Omega^{n+1} = 0$.*

Exercises 4.1.

1. Prove Lemma 4.1.
2. Prove Proposition 4.2.
3. Prove Lemmas 4.3 and 4.4.

4.2 Mabuchi energy

We next explain how the question of whether or not $\mathcal{M}$ contains a constant scalar curvature metric is encoded in a special function, called Mabuchi energy E. To define E, we choose a path $h_t = e^{frm-epi\phi_t} h_0$ of metrics in $\mathcal{M}$, where $\phi_t \in C^\infty(X, \mathbb{R})$ is a smooth path of Kähler potentials. In the following we write $\omega_t = \frac{i}{2\pi} F_{h_t}$

Lemma 4.5. *The quantity*

$$E(\omega_0; \omega_1) = \int_0^1 \int_X (S(\omega_t) - \bar{S})\, \dot{\phi}\, \frac{\omega_t^n}{n!}$$

depends only on the end points ω_0 and ω_1 and not on the path h_t joining h_0 to h_1.

Definition 4.6. The quantity $E(\omega_0; \omega_1)$ is called the *Mabuchi energy of ω_1 relative to ω_0.*

Fixing a reference metric ω_0, the function $E\colon \mathcal{H} \to \mathbb{R}$ defined by $E(\omega) = E(\omega_0; \omega_1)$ is simply called *Mabuchi energy*. Note that E depends on the choice of ω_0. Changing the reference metric will change E by a constant.

We also use the same notation for the function $E\colon \mathcal{M} \to \mathbb{R}$ defined by pulling back Mabuchi energy from $\mathcal{H} \to \mathbb{R}$ via the map $\mathcal{M} \to \mathcal{H}$ which sends $h \mapsto \frac{i}{2\pi}F_h$.

Mabuchi energy has the following important properties.

Proposition 4.7.

1. *The critical points of $E\colon \mathcal{M} \to \mathbb{R}$ are precisely those h for which ω_h has constant scalar curvature.*

2. *The Hessian of E at h is given by*

$$\mathcal{D}^*\mathcal{D}\colon C^\infty(X,\mathbb{R}) \to C^\infty(X,\mathbb{R})$$

where the operator $\mathcal{D}$ and its adjoint are computed with respect to ω_h.

It follows that E is convex along geodesics. Moreover, if there are no holomorphic vector fields on X which lift to L then E is strictly convex *along geodesics, except for those which correspond to scaling h by a constant.*

3. *Let v be a holomorphic vector field with lift $\hat{v}$ to L and write $f_t\colon L \to L$ for the flow of $\hat{v}$. Put $h_t = f_t^* h$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(h_t) = F(v)$$

One thing that is immediately suggested by this result is that, at least when there are no infinitesimal automorphisms of $L \to X$ outside of the scalars, there is at most one constant scalar curvature metric in $c_1(L)$ is unique.

To see why this should be the case, assume there were two such metrics $\omega_0, \omega_1 \in \mathcal{H}$. In finite dimensions, any two points in a negatively curved symmetric space are joined by a unique geodesic. In infinite dimensions this is no longer automatic—geodesics are the solutions to PDEs rather than ODEs and so their existence is more subtle. However, assuming for the moment that ω_0 and ω_1 are joined by a geodesic, the restriction of E to this geodesic is both strictly convex and has critical points at each ω_i. Hence we arrive at a contradiction unless $\omega_0 = \omega_1$. The hard part to making this argument rigorous is proving the existence of the geodesic. This has been done by X.-X. Chen, with sufficient regularity to carry through the above outline of a proof.

Exercises 4.2.

1. Prove Lemma 4.5.

2. Prove Proposition 4.7.

4.3 From geodesics to test configurations

The next thing that this result suggests is that it should be possible to ascertain whether or not there is a constant scalar curvature metric in $c_1(L)$ by looking at the behaviour of E at infinity. At least in finite dimensions, a convex function has a minimum if and only if it is proper, i.e., it tends to infinity at infinity. To investigate the behaviour of E at infinity, imagine picking a base point $h \in \mathcal{M}$ and a geodesic $\gamma\colon [0,\infty) \to \mathbb{R}$ starting at h and heading in the direction $u \in T_h\mathcal{M}$. Restricting E to the geodesic gives a convex function $f_u = E \circ \gamma\colon [0,\infty) \to \mathbb{R}$ which tends to infinity precisely when $\lim f'_u > 0$ as $t \to \infty$. In this way one is lead to the idea that the existence of a constant scalar curvature Kähler metric in $c_1(L)$ should be equivalent to $\lim f'_u > 0$ for all $u \in T_h\mathcal{M}$. (The problem with taking such a statement literally is that it presupposes the existence of geodesics leaving h in all directions and existing for all times, something which is known not to be true.)

The Yau–Tian–Donaldson conjecture has at heart the idea that the limits $\lim f'_u$ have a purely algebro-geometric interpretation, related to the Futaki invariant. To understand this, we first need to explain how to convert a path of Kähler metrics ω_t on a fixed complex manifold (X, J) to a path of complex structures J_t on a fixed symplectic manifold (X, ω). The key to this is the following lemma.

Lemma 4.8. *Given a function $\psi\colon X \to \mathbb{R}$ on a Kähler manifold (X, ω, J),*

$$i\bar{\partial}\partial\psi = L_{\nabla\psi}\omega$$

Because of this, given a path of Kähler metrics $\omega_t = \omega_0 + i\bar{\partial}\partial\phi$, we can define a path of vector fields, v_t by

$$v_t = \nabla_{\omega_t}\dot{\phi}$$

and integrate this to a path $f_t\colon X \to X$ of diffeomorphisms. By construction, $\omega_t = f_t^*\omega_0$ and so we can think of the path of metrics as being defined by a *fixed* symplectic form ω_0 and a path $J_t = (f_t^{-1})^*J$ of complex structures. The point here is that whilst for each finite t the complex structures J_t and J_0 are equivalent (they are related by the diffeomorphism f_t) in the limit, this need no longer be the case. One should imagine that, in the case ω_t is a geodesic, the complex manifolds (X, J_t) undergo a degeneration of some sort in the limit $t \to \infty$, whose behaviour encodes the derivative of E in this direction, in a sense to be made precise.

Recall above we interpreted a geodesic in $\mathcal{M}$ as a family of metrics on X parametrised by $\mathbb{C}^*$, (with trivial S^1-dependence). Switching point of view, we can think instead of a family $\mathcal{X}' \to \mathbb{C}^*$ of complex manifolds. Moreover, the path of diffeomorphisms generated by $v_t = \nabla_{\omega_t}\dot{\phi}$ gives a $\mathbb{C}^*$-action on $\mathcal{X}'$ covering the action by multiplication on the base $\mathbb{C}^*$. Changing coordinate $z \mapsto 1/z$ in $\mathbb{C}^*$, so that $t \to \infty$ corresponds to $z \to 0$, we see that our hoped for degeneration amounts to filling in the family $\mathcal{X}' \to \mathbb{C}^*$ to a family $\mathcal{X} \to \mathbb{C}$.

One situation in which this can be done explicitly is when the geodesic in $\mathcal{M}$ comes from a geodesic in G/K, where K is the isometry group of (X, J, ω). Such a geodesic corresponds to a 1-parameter subgroup $\mathbb{C}^* \subset G$ and hence a holomorphic vector field v on X. Tracing through the details, one finds that the family is holomorphically trivial $\mathcal{X} = X \times \mathbb{C}$, but with a non-trivial action, generated by $v + z\partial_z$. Notice that in this case $\lim f'_u$ is precisely the Futaki invariant of v, i.e., of the $\mathbb{C}^*$-action on the central fibre of $\mathcal{X}$.

Returning to the general discussion, we suppose family $\mathcal{X}' \to \mathbb{C}^*$ can be filled in to $\mathcal{X} \to \mathbb{C}$ in such a way that the $\mathbb{C}^*$-action extends to to $\mathcal{X}$ (just as happened for a geodesic arising from a holomorphic vector field on X). Then the action will necessarily fix the central fibre X_0 over $0 \in \mathbb{C}$. This means that one can take the Futaki invariant F of the action on X_0 and it is this which should correspond to $\lim f'_u$, just as was the case for a geodesic defined by a holomorphic vector field. (Note that in general the central fibre can be singular and so here we need to use the generalised Futaki invariant, which makes sense for $\mathbb{C}^*$-actions on polarised schemes. This also requires that the action lifts to the polarisation $L \to X$, which we have ignored in our above discussion.)

The above discussion is meant to be taken with a pinch of salt. It's main point is to motivate the following definitions.

Definition 4.9. Let $L \to X$ be a positive line bundle over a compact complex manifold. A *test configuration* for $L \to X$ is the following data:

1. A scheme $\mathcal{X}$, the total space of a flat family $\pi \colon \mathcal{X} \to \mathbb{C}$, together with a $\mathbb{C}^*$-action on $\mathcal{X}$, making π equivariant with respect to the action by multiplication on $\mathbb{C}$.

2. A polarisation $\mathcal{L} \to \mathcal{X}$ together with a lift of the $\mathbb{C}^*$-action to a linear action on $\mathcal{L}$.

3. An isomorphism between the fibre $L_1 \to X_1$ of $\mathcal{L} \to \mathcal{X}$ over $1 \in \mathbb{C}$ and $L^r \to X$, where r is a positive integer, called the *exponent* of the test configuration.

A *product configuration* is one of the form $L \times \mathbb{C} \to X \times \mathbb{C}$ with a product $\mathbb{C}^*$-action, namely one generated by $v + z\partial_z$, where v generates a $\mathbb{C}^*$-action on (L, X).

Definition 4.10. The*Futaki invariant* of a test configuration $(\mathcal{L}, \mathcal{X})$, is the Futaki invariant of the $\mathbb{C}^*$-action on the central fibre $L_0 \to \ X_0$ of $\mathcal{X}$ over $0 \in \mathbb{C}$.

Definition 4.11. A polarised complex manifold $L \to X$ is called *K-stable* if the Futaki invariant of every test configuration is non-negative and is equal to zero if and only if the configuration is a product.

For a while it was believed that K-stability was a necessary and sufficient condition for the existence of a constant scalar curvature metric in $c_1(L)$. Indeed this conjecture went by the name of the Yau–Tian–Donaldson conjecture. (Yau first suggested the existence of a Kähler–Einstein metric on a Fano manifold should be equivalent to "some notion of stability in the sense of geometric invariant theory". This was later refined to a precise statement by Tian, for Kähler–Einstein metrics and then Donaldson for metrics of constant scalar curvature.) However, recent developments have led to the realisation that for this to be true, the definition of K-stability given immediately above must be modified slightly.

The first development was an example found by Apostolov–Calderbank–Gauduchon–Tønnensen-Friedman, of a manifold which does not admit a constant scalar curvature metric and yet for which the obvious attempt to build a destabilising test configuration leads to a limit of test configurations in which one must take successively higher and higher exponents. Intuitively, one might think that the test configurations as described above probe a dense subset of the directions at infinity in $\mathcal{M}$, but to obtain information about all the directions, one should take limits of test configurations too. The second development was the discovery by Li and Xu that it is possible to build test configurations which are "trivial in codimension 2" but not products, which none-the-less have zero Futaki invariant. One should also adjust the definition to disregard these test configurations. An approach to both of these problems has been recently suggested by Székelyhidi. He embeds the space of test configurations in a larger ambient space—filtrations on the ring $\bigoplus H^0(X, L^k)$—where one can take limits. Filtrations have a natural norm and includes this norm together with the Futaki invariant in the definition of K-stability. This also seems to deal with the problem of Li and Xu's test configurations which have norm zero and so are automatically disregarded by the theory. Unfortunately we do not have the time here to go into the details of Székelyhidi's approach.

In one direction, and under certain hypotheses, the Yau–Tian–Donaldson conjecture is known to be true. Stoppa, building on substantial work by Tian and Donaldson, proved that when X admits no holomorphic vector fields and $c_1(L)$ contains a constant scalar curvature Kähler metric, then (X, L) is K-stable with respect to all test configurations which are non-trivial up to codimension 2. The converse direction is completely open (although Donaldson together with collaborators X.-X. Chen and S. Song have made a lot of progress recently in the Kähler–Einstein case, their plan of attack founded on earlier work of Tian).

5 Projective embeddings and the theorems of Kodaira and Tian

We now change subject and leave behind for a while the problem of finding canonical Kähler metrics. Instead we focus on one of the main sources of examples of Kähler metrics, namely projective geometry, and the complex submanifolds $X \subset \mathbb{CP}^N$. A natural question to ask is if a given Kähler manifold can be realised as a projective submanifold. If so, a second question is to ask how many Kähler metrics can be got via such embeddings and the restriction of the ambient Fubini–Study metric. We will address both these questions in this section.

5.1 Line bundles and maps to projective spaces

To construct a map from X to projective space we begin with a holomorphic line bundle $L \to X$ and a linear subspace $V \subset H^0(X, L)$ of holomorphic sections (which in later uses we will typically take to be the whole space). Such a V determines a map to projective space in the following way.

Let $s_0, \ldots, s_d$ be a basis of V and define the map $f\colon X \to \mathbb{CP}^d$ by

$$f(x) = [s_0(x) : \cdots : s_d(x)]$$

There are two things to mention here. Firstly, the $s_j(x)$ are not, as the notation here suggests, genuine complex numbers, rather they are all elements in the same complex line L_x, the fibre of L over $x \in X$. In order to make sense of the above expression, one must first choose an isomorphism $L_x \cong \mathbb{C}$, under which the $s_j(x) \in L_x$ are now identified with complex numbers $s_j'(x) \in \mathbb{C}$ say. The point is that if one chooses a different isomorphism between L_x and $\mathbb{C}$, the $s_j(x)$ become identified with different elements $s''_j(x) \in \mathbb{C}$ *but* since the two different identifications of L_x with $\mathbb{C}$ differ simply by multiplication by some $\alpha \in \mathbb{C} \setminus 0$, these new elements are related to the old ones by $s''_j(x) = \alpha s'_j(x)$ for all j and hence the corresponding point in projective space is unchanged. This is what is meant by the above map.

The second thing to say is that it is possible that f is not defined at all points of X, namely if all sections in V vanish at some x, then f will not be defined there.

Definition 5.1. Given a holomorphic line bundle $L \to X$ and a subspace $V \subset H^0(X, L)$, the set B of common zeros of sections of V is called the *base locus* of V. Given a basis of sections of V, $s_0, \ldots, s_d$, there is a well defined map $f \colon X \setminus B \to \mathbb{CP}^d$, called the *map corresponding to the linear system V.*

When $B = \varnothing$, one says that V is base point free.

Finally, when V is the whole space of sections, one calls V the *complete linear system of L.*

There is a more invariant way of defining the map f which does not involve the choice of a basis. To see this, notice that evaluation at a point $x \in X$ defines a linear map $\mathrm{ev}_x \colon V \to L_x$. Picking an identification $L_x \cong \mathbb{C}$ we identify ev_x with an element in V^*. Changing the identification $L_x \cong \mathbb{C}$ scales this element of V^* by a non-zero constant and so, at least assuming ev_x is not identically zero, we obtain a well-defined element of $\mathbb{P}(V^*)$.

Definition 5.2. Given a holomorphic line bundle $L \to X$ and a subspace $V \subset H^0(X, L)$ there is a canonically defined map, $f \colon X \setminus B \to \mathbb{P}(V^*)$, called the *map corresponding to the linear system V.*

When L is base point free, so that the map f is defined on all of X, one can recover the line bundle L from the map.

Lemma 5.3. *Given a line bundle $L \to X$ which is base point free, with corresponding map $f \colon X \to \mathbb{P}(H^0(X, L)^*)$, there is a natural identification between L and the pullback $f^*\mathcal{O}(1)$ of the hyperplane bundle.*

(Recall that the hyperplane bundle $\mathcal{O}(1) \to \mathbb{CP}^d$ is defined as the dual of the tautological bundle $\mathcal{O}(-1)$.)

Examples 5.4.

1. We begin with a tautological example. Recall that an element of $\mathcal{O}(-1)$ is a line in $\mathbb{C}^{d+1}$ together with a point on that line. From here it is easy to write down sections of $\mathcal{O}(1)$: any element s of the dual

vector space $(\mathbb{C}^{d+1})^*$ restricts to a linear map on each line in $\mathbb{C}^{d+1}$ and hence each fibre of $\mathcal{O}(-1)$, giving a holomorphic section of $\mathcal{O}(1)$. It is not too difficult to check that all holomorphic sections of $\mathcal{O}(1)$ arise this way. The map corresponding to the complete linear system $\mathbb{CP}^d \to \mathbb{P}((\mathbb{C}^{d+1})^*)^*)$ just amounts to the natural identification of the double dual with the original vector space.

2. More interesting examples are provided by taking powers $\mathcal{O}(1)^{\otimes k} = \mathcal{O}(k)$ of the hyperplane bundle. A holomorphic section of $\mathcal{O}(1)$ was just seen to be an element of $(\mathbb{C}^{d+1})^*$, i.e., a homogeneous linear polynomial in $n+1$ variables. In a similar way, a holomorphic section of $\mathcal{O}(k)$ is a homogeneous polynomial of degree k in $d+1$ variables. It can be checked that space of such polynomials has dimension $N_{k,d} = \frac{(k+d)!}{k!d!}$. Since there is no point of $\mathbb{C}^{d+1}$ at which all such polynomials vanish, the base locus of the complete linear system is empty and we get a map $\mathbb{CP}^d \to \mathbb{CP}^{N_{k,d}-1}$, called *the Veronese embedding*. It is not difficult to check that this is indeed an embedding.

Exercises 5.1.

1. Prove Lemma 5.3.

2. You will need to know the Riemann–Roch theorem on curves to do this question.

   (a) Prove that for a compact curve of genus at least 2, the complete linear system of the canonical bundle is base point free. In other words, there is no point at which all holomorphic 1-forms vanish.

   (b) From the previous part, we see that every compact curve Σ of genus at least 2 comes with a canonically defined map $\Sigma \to \mathbb{CP}^{g-1}$. Prove that one of two things happens. Either this map is an embedding, or it factors through a double cover $\Sigma \to \mathbb{CP}^1$ composed with the Veronese embedding $\mathbb{CP}^1 \to \mathbb{CP}^{g-1}$.

5.2 Kodaira's theorem on projective embeddings

Heuristically at least, the more holomorphic sections one has, the better the chances of the base locus vanishing or, even better, the corresponding map being an embedding. One way to increase the number of sections is to take powers of L. Every section s of L defines a section s^k of L^k, but in general one might hope that there are more sections of L^k than just these. We have just seen an example of this for $\mathcal{O}(1)$ and $\mathcal{O}(k)$.

Definition 5.5. A holomorphic line bundle $L \to X$ is called *very ample* if the complete linear system $H^0(X, L)$ defines an embedding of X into projective space.

A line bundle L is called *ample* if L^k is very ample for all large k.

Theorem 5.6 (Kodaira)**.** *A line bundle is ample if and only if it is positive.*

Recall that L is positive if it admits a positive Hermitian metric, i.e., one for which $\frac{i}{2\pi}F$ is a Kähler form. In one direction Kodaira's theorem is obvious: if $f \colon X \to \mathbb{CP}^d$ is a projective embedding, the pull back of the Fubini–Study metric on $f^*\mathcal{O}(1)$ is positively curved. So if L is ample, L^k is positive for some large k, and the k^{th} root of that positive metric is a positive metric in L. The hard part of the theorem is the converse, that positivity implies ampleness. We now sketch a proof of this.

The rough idea is that given $x \in X$, as k becomes large we can find holomorphic sections of L^k which are more and more concentrated at x. This means, in particular, there is a section which is non-zero there. Moreover, the sections concentrated near x and near y suffice to distinguish the images of x and y under the map to projective space.

More precisely we will sketch a proof of the following fact.

Theorem 5.7 (Existence of peaked sections)**.** *Let $x \in X$ and write $V_x \subset H^0(X, L^k)$ for the subspace of all sections vanishing at x.*

1. *For all large k, V_x has codimension 1.*
2. *Write $s_{k,x}$ for a generator of the L^2-orthogonal complement of V_x, with unit length in L^2. Then*
   (a) $|s_{k,x}(x)|^2 = k^n + O(k^{n-1})$
   (b) *for* $y \neq x$, $|s_{k,x}(y)| = O(k^{-\infty})$

(Here $O(k^{-\infty})$ means a quantity $f(k)$ which decays quicker than any polynomial).

Before outlining the proof of Theorem 5.7 let us sketch why this proves Kodaira's theorem. Firstly, the fact that V_x has codimension 1 is equivalent to saying that the base locus of L^k is empty, so we have a well defined map $X \to \mathbb{P}(H^0(X, L^k)^*)$. We next need to check that this is an embedding. We will settle for seeing that is an injection, namely that if x, y are distinct then there is a section which vanishes at x but not at y. To do this consider $s_{k,x}$ and $s_{k,y}$. Since $s_{k,x}(x) \neq 0$, we can find $a \in \mathbb{C}$ such that $as_{k,x} + s_{k,y}$ vanishes at x. But this section can't vanish at y since $|s_{y,k}|^2(y) = O(k^n)$ whilst $|s_{k,x}(y)| = O(k^{-\infty})$.

5.3 Existence of peaked sections

We now focus on the proof of Theorem 5.7. We will first produce a section $s'_{k,x}$ of L^k which has the properties listed in part 2. This will in particular imply part 1. The properties of part 2 essentially imply that $s'_{k,x}$ converges to $s_{k,x}$ in C^∞ as $k \to \infty$ from which it follows that this section also enjoys all the properties of part 2. We will thus concentrate just on producing a section $s'_{k,x}$ which satisfies the conclusions of part 2. (In fact, this is enough to prove Kodaira's theorem, we will only need the part about L^2-orthogonality later.)

We begin by considering the Euclidean case. We take for L the trivial bundle $\mathbb{C} \times \mathbb{C}^n$ together with the metric $h(z) = e^{-\pi|z|^2}$. This has curvature $F_h = -\pi\bar{\partial}\partial|z|^2 = \pi \sum \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. The corresponding real $(1,1)$-form is $\omega = \frac{i}{2\pi}F_h = \sum \mathrm{d}x_j \wedge \mathrm{d}y_j$, which is of course the standard flat metric on $\mathbb{C}^n$.

Now we consider L^k which is again, of course, trivial, but inherits the metric $h^k = e^{-k\pi|z|^2}$. In other words, the "constant" section, i.e., the section which takes the value 1 in the trivialisation of L^k, has length $e^{-k\pi|z|^2}$. We normalise this section by scaling it to have unit L^2-norm (with respect to the standard flat metric on $\mathbb{C}^n$). This gives, for each k, a section s_k of L^k whose point-wise norm is

$$|s_k(z)|^2 = k^n e^{-k\pi|z|^2}.$$

As $k \to \infty$, these Gaussian distributions converge to a Dirac delta centred at the origin. Notice that s_k certainly satisfies the conclusions of the theorem concerning peaked sections.

Next, return to the general case of a positively curved line bundle $L \to X$. Pick a point x and a small ball B containing it over which L is trivial. Over B, the geometry of $(X, L^k, h^k, k\omega)$ becomes closer and closer to the flat model (the metric $k\omega$ is close to flat when k is large). With this in mind we try to glue in the model peaked section from the above discussion. To do this we use a cut-off function in $\mathbb{C}^n$ and the resulting section $\tilde{s}_{k,x}$ of L^k is no longer holomorphic: it is holomorphic in the middle of B, zero outside of B and $\partial\tilde{s}_{k,x}$ is supported in an annulus in B. Moreover, because the Euclidean model agrees very closely with the geometry of $L^k \to X$ the "error" $\bar{\partial}\tilde{s}_{k,x}$ is small, in say L^2.

We now need to know how to correct this error and adjust $\tilde{s}_{k,x}$ to a genuine holomorphic section without destroying its "peaked" nature. We will solve

$$\bar{\partial} f_k = -\bar{\partial}\tilde{s}_{k,x}$$

and then set $s_{k,x} = \tilde{s}_{k,x} + f_k$. But of course we want f_k to be as small as possible (certainly not, for example just $-\tilde{s}_{k,x}$ which would leave us with the zero section!).

To do this we use something called "Hörmander's technique" which centres on the Bochner identity which we explain next. Recall that we defined the ∂- and $\bar{\partial}$-Laplacians on a Hermitian manifold and saw that when the metric was Kähler they were equal. We can do the same for forms with values in a holomorphic Hermitian vector bundle (E, h). The Chern connection ∇ splits as $\partial = \pi^{1,0} \circ \nabla$ and $\bar{\partial} = \pi^{0,1} \circ \nabla$. (This second of course does not depend on the choice of metric h, but the first operator does.) We write $\Lambda \colon \Omega^{p,q} \to \Omega^{p-1,q-1}$ for the adjoint to wedge product with ω.

Theorem 5.8 (Bochner, Kodaira, Nakano)**.** *Let $E \to X$ be a holomorphic Hermitian vector bundle over a Kähler manifold. Then the ∂- and $\bar{\partial}$-Laplacians on E-valued forms are related by*

$$\Delta_{\bar{\partial}} = \Delta_{\partial} + [iF, \Lambda]$$

where F is the curvature of the Chern connection in E.

This is proved via twisted versions of the Kähler identities, just as in the case of the two Laplacians acting on functions. At some point in the proof, one needs to commute two derivatives which explains the presence of the curvature F in the formula.

We will ultimately be interested in $(0, q)$-forms with values in L^k (such as $\bar{\partial}\tilde{s}_k$), but to get there via the Bochner–Kodaira–Nakano identity stated above we will use a trick and consider instead the line bundle $K^* \otimes L^k$. The point is that an (n, q)-form with values in $K^* \otimes L^k$ is the same thing as a q-form with values in L^k.

Now $K^* \otimes L^k$ has curvature

$$F = -2\pi i k\omega - i\rho$$

where ρ is the Ricci form of X. On (p, q)-forms, one checks directly that

$$[\omega, \Lambda] = p + q - n$$

where $n = \dim X$. It follows that on (n, q)-forms with values in $K^* \otimes L^k$, or equivalently, on $(0, q)$-forms with values in L^k,

$$\Delta_{\bar{\partial}} = \Delta_{\partial} + 2\pi q k + [\rho, \Lambda].$$

Now Δ_{∂} is semi-positive and $[\rho, \Lambda]$ is independent of k. Hence there is a constant C such that for all $f \in \Omega^{0,q}(X, L^k)$,

$$\langle \Delta_{\bar{\partial}} f, f \rangle_{L^2} \geq (2\pi q k - C) \, \|f\|^2_{L^2}$$

This is the fundamental inequality with the following immediate consequences

Theorem 5.9 (Kodaira vanishing and the spectral gap)**.** *Let $L \to X$ be a positive line bundle. There is a constant C such that for all $q > 0$ and all sufficiently large k, the lowest eigenvalue ν of $\Delta_{\bar{\partial}}$ acting on $\Omega^{0,q}(X, L^k)$ satisfies $\nu \geq 2\pi qk - C$.*

In particular $\Delta_{\bar{\partial}}$ is invertible for large k and hence $H^q(X, L^k) = 0$ for all $q > 0$. (This is known as Kodaira's vanishing theorem.)

Moreover, the first non-zero eigenvalue μ of the operator $\Delta_{\bar{\partial}}$ acting on sections of L^k satisfies $\mu \geq 2\pi k - C$.

The bound on ν follows from that on μ since if $\Delta_{\bar{\partial}} f = \lambda f$ for $\lambda \neq 0$ and $f \in \Omega^0(X, L^k)$, then $\bar{\partial} f \in \Omega^{0,1}(X, L^k)$ is non-zero and so again an eigenvector of $\Delta_{\bar{\partial}}$ with eigenvalue λ.

From here we can deduce Hörmander's estimates for solutions of the $\bar{\partial}$-equation:

Theorem 5.10 (Hörmander's estimate)**.** *For all large k, given $g \in \Omega^{0,1}(X, L^k)$ with $\bar{\partial} g = 0$ then there is a section $f \in \Omega^0(X, L^k)$ such that*

$$\bar{\partial} f = g$$

Moreover there is a constant C, independent of g such that the above solution satisfies $\|f\|_{L^2} \leq Ck^{-1}\|g\|_{L^2}$.

To see this note that $\bar{\partial}^* g$ is automatically orthogonal to $\ker \partial$ which is precisely where we can invert $\Delta_{\bar{\partial}}$. Set $f = \Delta_{\bar{\partial}}^{-1}(\bar{\partial}^* g)$. Then $\bar{\partial} f = g$ since $\bar{\partial} g = 0$ implies that $\Delta_{\bar{\partial}} g = \bar{\partial}\bar{\partial}^* g$. Finally the estimate on $\|f\|_{L^2}$ follows from the lower bound on the first non-zero eigenvalue of $\Delta_{\bar{\partial}}$ on sections proved above.

Return now to our goal of producing a section $s'_{k,x}$ of L^k peaked at a point x, in the sense that it has all the properties listed in part 2 of Theorem5.7. Recall that we began by gluing in a peaked section using the Euclidean model to obtain a section $\tilde{s}_{k,x}$ with $\|\bar{\partial}\tilde{s}_{k,x}\|_{L^2} = O(1)$. Now apply Hörmander's estimate to obtain a solution to $\bar{\partial} f_k = -\bar{\partial}\tilde{s}_{k,x}$ with $\|f_k\|_{L^2} \leq Ck^{-1}$. Setting $s'_{k,x} = \tilde{s}_{k,x} + f_k$ we obtain a holomorphic section of L^k which is very close to the glued in Gaussian when k is large, at least initially L^2. To get better control of the adjustment in f_k one needs to use standard elliptic estimates for $\Delta_{\bar{\partial}}$ to pass from L^2 to C^k. We do not give the details here.

Now $s'_{k,x}$ is non-zero at x (it is of order k^n even) and so the subspace $V_x \subset H^0(X, L^k)$ of sections vanishing at x is indeed of codimension 1.

Moreover, whilst $s'_{k,x}$ is not quite L^2-orthogonal to V_x it is asymptotically so as $k \to \infty$, because it's mass in L^2 is localised at x. From here one can finish the proof of Theorem 5.7 by projecting $s'_{k,x}$ to $V_x^{\perp}$.

Exercises 5.2.

1. Let $E \to X$ be a Hermitian holomorphic vector bundle over a Kähler manifold. We write ∂_E and $\bar{\partial}_E$ for the $(1,0)$ and $(0,1)$-components respectively of the Chern connection on E. We also write $L(\alpha) = \omega \wedge \alpha$ for the operation of wedging with the Kähler form.

   Prove the twisted Kähler identities:

$$[\partial_E^*, L] = -i\bar{\partial}_E, \qquad [\bar{\partial}_E^*, L] = i\partial_E$$

2. Starting from the twisted Kähler identities, prove the Bochner–Kodaira–Nakano identity, Theorem 5.8 above.

3. Recall $L\colon \Omega^{p,q} \to \Omega^{p+1,q+1}$ is the operation of wedging with ω, whilst $\Lambda\colon \Omega^{p,q} \to \Omega^{p-1,q-1}$ is its adjoint.

   Prove that on (p,q)-forms $[L, \Lambda] = p + q - n$.

4. Prove that if L is a positive line bundle and $p + q > n$ then $H^{p,q}(X, L) = 0$. (This is called Nakano's vanishing theorem.)

5.4 Tian's theorem on projective embeddings

Let $L \to X$ be a positive line bundle. We are interested in the space $\mathcal{H}$ of all Kähler metrics in $c_1(L)$. By Kodaira's theorem, high powers L^k give rise to embeddings into projective spaces $\mathbb{P}(H^0(X, L^k)^*)$. If we choose a basis of $H^0(X, L^k)$ we can identify with a "standard" projective space $\mathbb{CP}^{d_k}$ and pull the Fubini–Study metric. This gives a metric $\frac{1}{k} f^*\omega_{\mathrm{FS}} \in c_1(L)$. (The rescaling is necessary since the unscaled metric lies in $c_1(L^k) = kc_1(L)$).

Varying the basis will, in general, give different metrics. The linear group $\mathrm{GL}(d_k + 1, \mathbb{C})$ acts transitively on the set of all bases and two choices determine the same metric if and only if they are related by an element of $\mathrm{U}(d_k + 1)$. It follows that using embeddings via L^k to produce metrics in yields a subset $\mathcal{B}_k \subset \mathcal{H}$,

$$\mathcal{B}_k \cong \mathrm{GL}(d_k + 1, \mathbb{C}) / \mathrm{U}(d_k + 1)$$

where $d_k + 1 = \dim H^0(X, L^k)$.

Definition 5.11. The subset $\mathcal{B}_k \subset \mathcal{H}$ is called the k^{th} Bergman space and its element are called *Bergman metrics at level k*.

A natural question is whether or not the Bergman spaces fill out all of $\mathcal{H}$ in the limit as $k \to \infty$. This is part of the content of Tian's theorem, which we will state shortly. In fact the theorem says more, given $\omega \in \mathcal{H}$, it gives a systematic way to construct a sequence $\omega_k \in \mathcal{B}_k$ of Bergman metrics which converge to ω as $k \to \infty$.

To construct ω_k, first let h be a Hermitian metric in L with curvature $F = -2\pi i\omega$. (This determines h up to multiplication by a constant, which will not change the end result.) Each space of sections $H^0(X, L^k)$ comes with an L^2-innerproduct. Choosing an orthonormal basis gives a projective embedding and hence a metric ω_k got by rescaling the restriction of the Fubini–Study metric. Choosing a different orthonormal basis corresponds to a unitary transformation of projective space which doesn't change the resulting metric.

So there is a canonical sequence $\omega_k \in \mathcal{B}_k$ associated to any point $\omega \in \mathcal{H}$.

Theorem 5.12 (Tian)**.** *Given any $\omega \in \mathcal{H}$, $\omega_k \to \omega$ as $k \to \infty$.*

To prove this we first introduce something called the Bergman function, $\beta_k \colon X \to \mathbb{R}$. For each k, let $s_0, \dots s_{d_k}$ be an orthonormal basis for $H^0(X, L^k)$. Then set

$$\beta_k(x) = \sum_{j=0}^{d_k} |s_j(x)|^2$$

One checks that this function does not depend on the choice of scale for h nor on the choice of orthonormal basis. It depends solely on ω and k. It can be thought of as a measure of how spread out the sections of L^k are over the manifold. The interest for us is that β_k determines the difference of ω and ω_k:

Lemma 5.13. $\omega_k = \omega + \frac{i}{k}\bar{\partial}\partial \log \beta_k$

This is a simple calculation based on the definition of the Fubini–Study metric. From here we see that Tian's theorem amounts to the statement that β_k is asymptotically constant. But in fact, we have (more-or-less!) proved this already during our discussion of Kodaira's theorem.

Theorem 5.14 (Tian)**.** *The function β_k has the property that*

$$\beta_k(x) = k^n + O(k^{n-1})$$

as $k \to \infty$. More precisely, for any r there is a constant C such that

$$\|1 - k^{-n}\beta_k\|_{C^r} \leq Ck^{-1}$$

for all large k.

To see why this should be true, pick $x \in X$ and let the first element s_0 in the basis be the section peaked at x provided by Theorem 5.7. Since the L^2-orthogonal space to s_0 consists of sections vanishing at x, we have that $\beta_k(x) = |s_0(x)|^2 = k^n + O(k^{n-1})$. (We admittedly haven't been precise enough in our discussion above to see that this holds in C^r.)

Now $\log \beta_k = n \log k + \log(k^{-n}\beta_k)$ and so $\| \log \beta_k - n \log k\|_{C^r} \leq Ck^{-1}$ for some constant C. From here it follows that

$$\|\omega_k - \omega\|_{C^{r-2}} \leq Ck^{-2}$$

which implies Tian's theorem.

Exercises 5.3.

1. Prove Lemma 5.13.

6 Balanced embeddings and Luo–Zhang's theorem

In this section we will discuss "best" projective embeddings which are projectively equivalent to a given one $X \subset \mathbb{CP}^d$. We will approach this in such a way as to highlight as much as possible the analogies with Calabi's suggestion of extremal metrics being best representatives of a given Kähler class.

6.1 Balanced embeddings and balancing energy

Throughout this and subsequent sections we will make use of an embedding $\mu \colon \mathbb{CP}^d \to \mathrm{Herm}(d+1)$ of projective space into the Euclidean space of Hermitian $(d+1) \times (d_1)$ matrices. We think of $\mathrm{Herm}(d+1)$ as a Euclidean vector space via the inner-product $(A, B) = \mathrm{Tr}(AB)$. To define μ we send a point $p \in \mathbb{CP}^n$ to the endomorphism of $\mathbb{C}^{n+1}$ which is orthogonal projection onto the line corresponding to p. It is straightforward to check that this is equivariant with respect to $\mathrm{U}(d+1)$. This means that the Euclidean metric on $\mathrm{Herm}(d+1)$ restricts to give a $\mathrm{U}(d+1)$-invariant metric on $\mathbb{CP}^d$ and this is one way of defining the Fubini–Study metric. In fact, if we identify $i\,\mathrm{Herm}(d+1)$ and $\mathfrak{u}(d+1)^*$ via the inner-product it is not hard to see that the map μ is essentially the moment map for the action of $\mathrm{U}(n+1)$ on $\mathbb{CP}^d$, embedding it as a coadjoint orbit.

Now, given a complex submanifold $X \subset \mathbb{CP}^d$ we can think of X as a subset of $\mathrm{Herm}(d+1)$ and ask for its centre of mass. We set

$$\bar{\mu}(X) = \int_X \mu \, \frac{\omega_{\mathrm{FS}}^n}{n!}$$