Riemann-Roch-Grothendieck theorem for families of curves with hyperbolic cusps and its applications to the moduli space of curves

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Plan of the talk

Riemann-Roch-Grothendieck theorem and curvature theorem

Family setting

$\pi : X \to S$ proper holomorphic submersion, relative dimension 1

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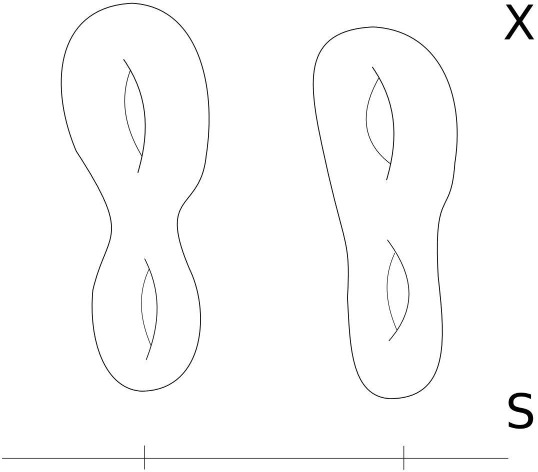
$$\omega_{X/S} = (\Lambda^{\max} T^{*(1,0)} X) \otimes (\Lambda^{\max} T^{*(1,0)} S)^{-1}$$

the relative canonical line bundle of π

$$t \in S, X_t = \pi^{-1}(t)$$

A picture



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$$H^0(X_t, \xi) = \ker(\overline{\partial}), \qquad H^1(X_t, \xi) = \Omega^{0,1}(X_t, \xi) / \operatorname{Im}(\overline{\partial})$$

The determinant of the cohomology

$$\lambda(j^*\xi)_t = (\Lambda^{\max} H^0(X_t, \xi|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, \xi|_{X_t}), \quad t \in S$$

family of complex lines over S

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Grothendieck-Knudsen-Mumford :
$\lambda(j^*\xi)_t$, $t \in S$ form a holomorphic line bundle $\lambda(j^*\xi)$ over S

Theorem of Riemann-Roch-Grothendieck

Theorem. (Riemann-Roch-Grothendieck, 1957)

The following identity holds in $H^\bullet(S, \mathbb{Q})$:

$$c_1(\lambda(j^*\xi)) = -\int_\pi \left[\mathrm{Td}(\omega_{X/S})\mathrm{ch}(\xi)\right]^{[4]}$$

$$\mathrm{Td}(\xi) = 1 + \frac{c_1(\xi)}{2} + \frac{c_1(\xi)^2 + c_2(\xi)}{12} + \dots$$

$$\mathrm{ch}(\xi) = \mathrm{rk}(\xi) + c_1(\xi) + \frac{c_1(\xi)^2 - 2c_2(\xi)}{2} + \dots$$

Chern-Weil theory

- Y a complex manifold

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- Chern-Weil : $\left[\mathrm{ch}(E, h^E)\right]_{DR} = \mathrm{ch}(E) \in \oplus_{p\in\mathbb{N}} H^{2p}(Y, \mathbb{R})$
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$$c_1(\lambda(j^* \xi), \textbf{?}) = - \int_{\pi} \Big[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|^{\omega}_{X/S})^2) \mathrm{ch}(\xi, h^{\xi}) \Big]^{[4]}$$

L^2 product and Hodge theory

- L^2-Hermitian product. Let $\alpha, \alpha' \in \Omega^{0,\bullet}(X_t, \xi)$
  $\left\langle \alpha, \alpha' \right\rangle_{L^2} = \int_{X_t} \left\langle \alpha(x), \alpha'(x) \right\rangle_h dv_{X_t}(x)$,
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$$\ker(\Box^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) \to H^\bullet(X_t, \xi)$$

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Hodge theory

- induces the L^2-norm $\|\cdot\|_{L^2}\left(g^{TX_t}, h^\xi\right)$ over
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Problem : Need to make sense of the infinite product...

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Definition of the determinant. (Ray-Singer, 1973)

$$\det{}' \Box_t^{\xi} = \exp\Big(- \zeta'_{\xi,t}(0) \Big)$$

Quillen norm

Hermitian norm on $\lambda(j^* \xi)$, given by

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$$c_1\left(\lambda(j^*\xi), \left(\|\cdot\|^Q\left(g^{TX_t}, h^\xi\right)\right)^2\right) = -\int_\pi \left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|^\omega_{X/S})^2)\mathrm{ch}(\xi, h^\xi)\right]^{[4]}$$

Motivation

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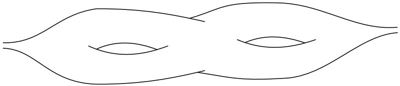
We want to extend the theory of Quillen metrics to surfaces with hyperbolic cusps and degenerating families with singular fibers

What is a surface with hyperbolic cusps ?



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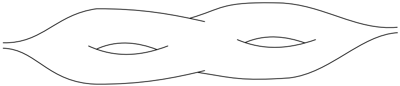

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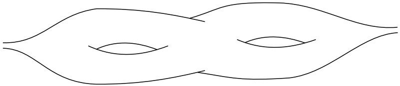

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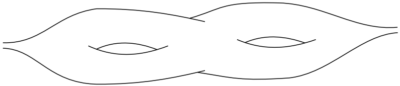

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We call $(\overline{M}, D_M, g^{TM})$ a surface with cusps

An important example

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- Arithmetic Riemann-Roch theorem for pointed stable curves

Definition of Quillen metric for surfaces with cusps

The L^2-norm

$$\|\cdot\|^{Q} = \left(\det{}'\Box\right)^{1/2} \cdot \|\cdot\|_{L^2}$$

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- For $n \leq 0$, by Hodge theory*
  $\langle \cdot, \cdot \rangle_{L^2}$ induces the L^2-norm $\|\cdot\|_{L^2}$ on
  $\lambda(E_n^{\xi}) = (\Lambda^{\max} H^0(\overline{M}, E_n^{\xi}))^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, E_n^{\xi})$

The determinant

$$\|\cdot\|^{Q} = \left(\det{}'\Box\right)^{1/2} \cdot \|\cdot\|_{L^2}$$

Problem with the determinant

$$\square^{E_n^{\xi}} : \Omega^{0,0}(M, E_n^{\xi}) \to \Omega^{0,0}(M, E_n^{\xi})$$

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$$Z_{(\overline{M}, D_M)}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})$$

γ simple closed geodesics on M ; $l(\gamma)$ is the length of γ.

Takhtajan-Zograf approach

Takhtajan-Zograf definition using Selberg zeta-function, 1991

$$\det{}'_{TZ} \Box^{E_n^\xi} = \begin{cases} Z'_{(\overline{M}, D_M)}(1), & \text{for } n = 0, \\ Z_{(\overline{M}, D_M)}(-n+1), & \text{for } n < 0. \end{cases}$$

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Limitations of this approach

- Restriction on the topology $2g(\overline{M}) - 2 + \#D_M > 0$.
- Complex structure predefines the Kähler metric.
- No liberty in choosing (ξ, h^{ξ}).

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$$\zeta_{E_n^\xi}(s) = \sum_{\lambda \in \mathrm{Spec}(\Box^{E_n^\xi}) \setminus \{0\}} \lambda^{-s} \tag{$\star$}$$

$$= \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}\Big[\exp^{\perp}(-t\Box^{E_n^\xi}) \Big] t^{s-1} dt \tag{$\star\star$}$$

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- For $m > 0$?
  Idea : define $\zeta_{E_n^\xi}(s)$ for $m > 0$ using $(\star\star)$ and not $(\star)$
- Problem : $\exp^{\perp}(-t\square^{E_n^\xi})$ is not of trace class for $m > 0$

$A \in \mathrm{End}(\mathbb{C}^n), n \in \mathbb{N},$ self-adjoint

Two faces of the trace

$$A \in \mathrm{End}(\mathbb{C}^n), n \in \mathbb{N}, \qquad \text{self-adjoint}$$

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Idea : if $\mathrm{Tr}\Big[A\Big] = +\infty$, we define $\mathrm{Tr}^r\Big[A\Big] = \sum_{a_{ii} \neq +\infty} a_{ii}$.

Regularizing trace, I

- The operator $\exp(-t\Box^{E_n^\xi})$ has a smooth Schwartz kernel

$$\exp(-t\Box^{E_n^\xi})(x,y) \in (E_n^\xi)_x \otimes (E_n^\xi)_y^*, \qquad x, y \in M$$

$$\exp(-t\Box^{E_n^\xi})s = \int_M \Big\langle \exp(-t\Box^{E_n^\xi})(x,y), s(y) \Big\rangle dv_M(y).$$

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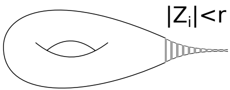
- Idea : define $\mathrm{Tr}^{\boldsymbol{r}}\Big[\exp(-t\Box^{E_n^\xi})\Big]$ by taking the finite part of

$$\int_{M_r} \mathrm{Tr}\Big[\exp(-t\Box^{E_n^\xi})(x,x)\Big] dv_M(x)$$

as $r \to 0$, where M_r is the non-striped region



$$P = \mathbb{C}P^1 \setminus \{0, 1, \infty\},$$
$$g^{TP} \text{ hyperbolic metric csc } -1 \text{ over } P$$

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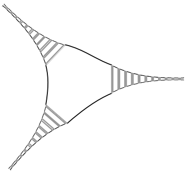
g^{TP} hyperbolic metric csc -1 over P

We fix $n \leq 0$

$$g_n(r, t) = \frac{1}{3} \int_{P_r} \exp(-t \square^{\omega_P(D)^n})(x, x) dv_P(x), \tag{4.1}$$

where P_r is the non-striped region



Theorem. (-, 2018)

For any $(\overline{M}, D_M, g^{TM})$, (ξ, h^{ξ}), $t > 0$, the function

$$\mathbb{R}_{>0} \ni r \mapsto \int_{M_r} \mathrm{Tr}\Big[\exp(-t\Box^{E_n^{\xi}})(x,x) \Big] dv_M(x) - \mathrm{rk}(\xi) \cdot m \cdot g_n(r,t)$$

extends continuously over $r = 0$.

Regularized heat trace

$$\mathrm{Tr}^{\boldsymbol{r}}\Big[\exp(-t\Box^{E_n^\xi})\Big] = \lim_{r\to 0}\Bigg(\int_{M_r}\mathrm{Tr}\Big[\exp(-t\Box^{E_n^\xi})(x,x)\Big]dv_M(x) - \mathrm{rk}(\xi)\cdot m\cdot g_n(r,t)\Bigg).$$

Regularized zeta function

$$\zeta_{E_n^\xi}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}^{\boldsymbol{r}}\left[\exp^{\perp}(-t\Box^{E_n^\xi})\right] t^{s-1} dt.$$

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Theorem. (-, 2018)

- $\zeta_{E_n^\xi}(s)$ is well-defined and extends meromorphically to $\mathbb{C}$
- $0 \in \mathbb{C}$ is a holomorphic point of $\zeta_{E_n^\xi}(s)$

Finally, the determinant

Definition of the determinant

$$\det{}' \Box^{E_n^{\xi}} = \exp\Big(- \zeta'_{E_n^{\xi}}(0) \Big).$$

Theorem. (-, 2019)

Suppose $(M, D_M, g^{TM}_{\mathrm{hyp}})$ has csc -1, (ξ, h^{ξ}) trivial. Then for any $m \geq 0$, $n \leq 0$, we have

$$\det{}' \Box^{E_n^{\xi}} =^* \det{}'_{TZ} \Box^{E_n^{\xi}}.$$

$=^*$ means up to some computed universal constant

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$m = 0$, Phong-D'Hoker, 1986

Finally, the Quillen norm

Quillen norm

Hermitian norm on $\lambda(E_n^\xi)$, given by

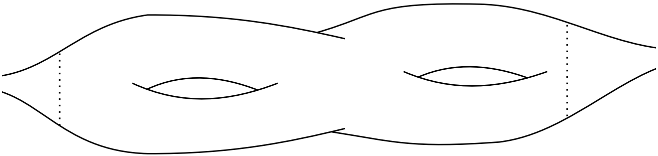
$$\left\|\cdot\right\|^{Q}\left(g^{TM}, h^{E_n^\xi}\right) = \left(\det{}'\square^{E_n^\xi}\right)^{1/2} \cdot \left\|\cdot\right\|_{L^2}\left(g^{TM}, h^{E_n^\xi}\right)$$

A question

How to compute the Quillen norm ?

Relative compact perturbation theorem

Flattening of a metric with cusps g^{TM}



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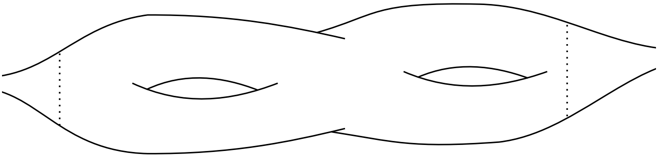
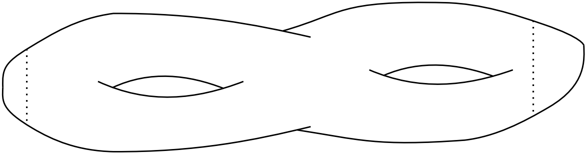


is a Kähler metric g_{f}^{TM} on $\overline{M}$ such that



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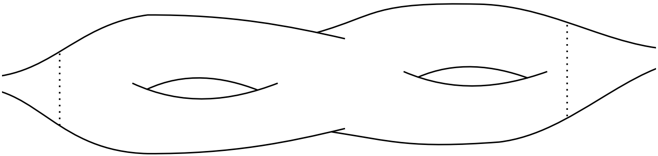

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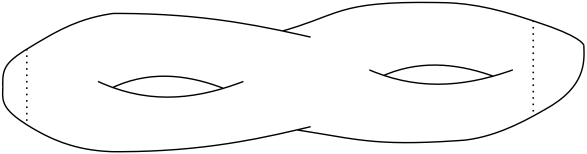

The same for $\|\cdot\|_M$

A problem

Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps,

(ξ, h^{ξ}) Hermitian vector bundle over $\overline{M}$

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We want to understand how to calculate

$$\frac{\|\cdot\|_Q \left(g^{TM}, h^{\xi} \otimes \|\cdot\|^{2n}_M\right)}{\|\cdot\|_Q \left(g^{TM}_{\mathrm{f}}, h^{\xi} \otimes (\|\cdot\|^{\mathrm{f}}_M)^{2n}\right)}$$

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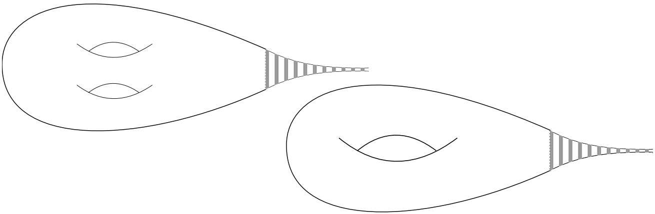
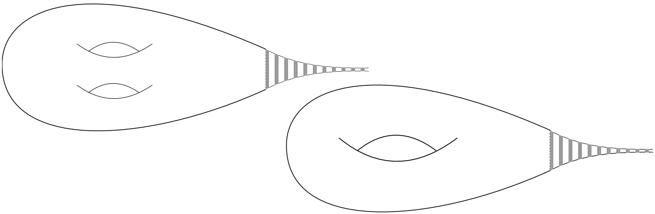
In other words :

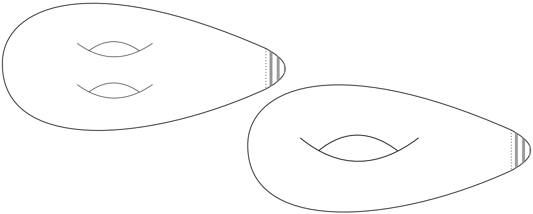
How Quillen metric changes under compact perturbation ?

Compatible flattenings





Two flattenings g_{f}^{TM}, g_{f}^{TN} of g^{TM}, g^{TN} are called compatible if



Relative compact perturbation theorem

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Theorem. (-, 2018)

For simplicity, suppose (ξ, h^ξ) is trivial

$$\frac{\|\cdot\|_Q \left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\right)} = \left(\frac{\|\cdot\|_Q \left(g^{TN}, \|\cdot\|_N^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TN}, (\|\cdot\|_N^{\mathrm{f}})^{2n}\right)}\right)^{\mathrm{rk}(\xi)}$$

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Theorem. (-, 2018)

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$$\cdot \exp\left(\frac{1}{2} \int_M c_1(\xi, h^\xi) \Big(2n \ln(\|\cdot\|^{\mathrm{f}}_M / \|\cdot\|_M) + \ln(g^{TM}_{\mathrm{f}} / g^{TM}) \Big) \right)$$

Anomaly formula

The Wolpert norm

$(\overline{M}, D_M, g^{TM})$, $D_M = \{P_1, \ldots, P_m\}$ surface with cusps

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Wolpert norm is related to the “constant term"
of the conformal transformation at cusp

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g^{TM}, g_0^{TM} metrics with cusps at D_M

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ξ holomorphic vector bundle on $\overline{M}$
h^ξ, h_0^ξ Hermitian metrics on ξ over $\overline{M}$

Anomaly formula

Theorem. (-, 2018)

$$2\log\Big(\|\cdot\|_Q \,(g_0^{TM}, h_0^{\xi} \otimes (\|\cdot\|_M^0)^{2n}) \Big/ \|\cdot\|_Q \,(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}) \Big)$$

$$= \int_M \Big[\text{Bott-Chern terms, analogic to the anomaly for compact manifolds of Bismut-Gillet-Soulé}\Big]$$

$$- \frac{\mathrm{rk}(\xi)}{6} \log\left(\|\cdot\|^W / \|\cdot\|_0^W \right) + \sum \log\left(\det(h^{\xi}/h_0^{\xi})|_{P_i} \right).$$

What is a family of curves with cusps ?

Family of curves with cusps

- $\pi : X \to S$ proper holomorphic of relative dimension 1, $t \in S$, $X_t = \pi^{-1}(t)$ has at most double-point singularities (i.e. those of the form $\{z_0 z_1 = 0\}$)

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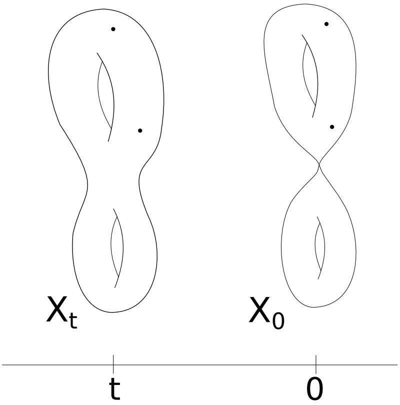
  $(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$ a family of curves with cusps

A picture



Curvature theorem for family of curves with cusps

$(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$ a family of curves with cusps

$\omega_{X/S}(D) = \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S}),$ $\|\cdot\|_{X/S}$

twisted relative canonical line bundle on X

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$$E_n^{\xi} = \xi \otimes \omega_{X/S}(D)^n$$

$$\lambda(j^* E_n^{\xi})_t = (\Lambda^{\max} H^0(X_t, E_n^{\xi}|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, E_n^{\xi}|_{X_t})$$

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Grothendieck-Knudsen-Mumford

$\lambda(j^* E_n^{\xi})_t$, $t \in S$ form a holomorphic line bundle $\lambda(j^* E_n^{\xi})$ over S

Quillen norm

We define the Quillen norm on $\lambda(\xi \otimes \omega_{X/S}(D)^n)$ by

$$\|\cdot\|^{Q}\left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}\right) = \left(\det{}'\square_t^{E_n^{\xi}}\right)^{1/2} \cdot \|\cdot\|_{L^2}\left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}\right).$$

Wolpert norm

We define the Wolpert norm $\|\cdot\|^W$ on $\otimes_i \sigma_i^*(\omega_{X/S})$ over S by gluing the Wolpert norms $\|\cdot\|_t^W$ on $\otimes_i \omega_{X/S}|_{\sigma_i(t)}$ induced by g^{TX_t}.

Necessity of additional hypothesis

We are in the non-compact setting !

Hypothesis on the metric

We suppose that the metric $\|\cdot\|_{X/S}$ induced on $\omega_{X/S}(D)$ is pre-log-log on X with singularities along $\pi^{-1}(|\Delta|) \cup D_{X/S}$

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If $\{z = 0\}$ is a local equation for $\pi^{-1}(|\Delta|) \cup D_{X/S}$ around a smooth point

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$$\partial \log(\|v\|_{X/S}) = O\Big((\log|\log|z||)^N \frac{dz}{z\log|z|}\Big)$$

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Wolpert, 1990, (compact case) and Freixas, 2007, (pointed case) proved : the metric of csc -1 on the relative twisted canonical line bundle of universal curve is good

Riemann-Roch-Grothendieck theorem in the presence of cusps

$$\mathscr{L}_n = \lambda(j^* E_n^{\xi})^{12} \otimes (\otimes_i \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes (\otimes_i \sigma_i^* \det \xi)^6$$

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Canonical singular norm

s_Δ the canonical holomorphic section of $\mathscr{O}_S(\Delta)$

$\|\cdot\|_\Delta^{\mathrm{div}}$ on $\mathscr{O}_S(\Delta)$ is defined by $\|s_\Delta\|_\Delta^{\mathrm{div}}(x) = 1, \qquad x \in S \setminus |\Delta|$

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Theorem. (-, 2018)

$\|\cdot\|^{\mathscr{L}_n}$ extends continuously* over $|\Delta|$, smooth* over $S \setminus |\Delta|$

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Theorem. (-, 2018)

$\|\cdot\|^{\mathscr{L}_n}$ extends continuously* over $|\Delta|$, smooth* over $S \setminus |\Delta|$, and on the level of currents over S :

$$c_1\left(\mathscr{L}_n, (\|\cdot\|^{\mathscr{L}_n})^2\right) = -12 \int_\pi \left[\mathrm{Td}(\omega_{X/S}(D), \|\cdot\|_{X/S}^2)\mathrm{ch}(\xi, h^{\xi})\mathrm{ch}(\omega_{X/S}(D), \|\cdot\|_{X/S}^{2n})\right]^{[4]}$$

Thank you !