Geometric quantization on CR manifolds

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Intrduction

- In this work we study the problem of "quantization commutes with reduction" for CR manifolds (for some non-hypoelliptic operators), in particular for Sasakian manifolds.
- An important difference between CR setting and symplectic setting is that the quantum spaces we considered in CR setting are infinite dimensional.

CR manifolds

- Let X be a smooth and orientable manifold of dimension $2n+1$, $n \geq 1$.
- Let $T^{1,0}X$ be a subbundle of $\mathbb{C}TX$ the complexified tangent bundle of X.

Definition

We say that $T^{1,0}X$ is a CR structure of X if

- (i) $\dim_{\mathbb{C}} T_x^{1,0}X = n$, for every $x \in X$.
- (ii) $T^{1,0}X \bigcap T^{0,1}X = \{0\}$, where $T^{0,1}X := \overline{T^{1,0}X}$.
- (iii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, $\mathcal{V} = \mathscr{C}^\infty(X, T^{1,0}X)$.

- For a $2n+1$ dimensional smooth manifold X, if we can find a CR structure $T^{1,0}X$ on X, we call the pair $(X, T^{1,0}X)$ a CR manifold.

CR manifolds

- Take a Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ such that we have the orthogonal decompositions:
  - $\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T$, $T \in \mathscr{C}^\infty(X, TX)$, $\|T\| = 1$,
  - $\mathbb{C}T^*X = T^{*1,0}X \oplus T^{*0,1}X \oplus \mathbb{C}\omega_0$, $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$, $\|\omega_0\| = 1$,
  - $\langle \omega_0 \,,\, T \rangle = -1$, $T^{*0,1}X = (T^{1,0}X \oplus \mathbb{C}T)^\perp$.
  - ω_0: Reeb one form, T: Reeb vector field, $T^{*0,1}X$: bundle of $(0,1)$ forms.

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Definition

For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T_p^{1,0}X$ given by $\mathcal{L}_p(U, V) = -\frac{1}{2i} d\omega_0(p)(U, \overline{V})$, $U, V \in T_p^{1,0}X$.

- We say that X is strongly psudoconvex at $p \in X$ if the Levi form is positive definite at $p \in X$.
- We say that X is strongly psudoconvex if the Levi form is positive definite at each point of X.

CR functions

- Let $\tau : \mathbb{C}T^*X \to T^{*0,1}X$ be the orthogonal projection.
- $\overline{\partial}_b = \tau \circ d : \mathscr{C}^\infty(X) \to \Omega^{0,1}(X)$: tangential Cauchy-Riemann(CR) operator, where $\Omega^{0,1}(X) = \mathscr{C}^\infty(X, T^{*0,1}X)$.

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- We extend $\overline{\partial}_b$ to L^2 space:
  $\overline{\partial}_b : \mathrm{Dom}\,\overline{\partial}_b \subset L^2(X) \to L^2_{(0,1)}(X)$, where
  $\mathrm{Dom}\,\overline{\partial}_b = \left\{u \in L^2(X); \overline{\partial}_b u \in L^2(X)\right\}$.

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- For a function $u \in L^2(X)$, we say that u is a CR function if $u \in \mathrm{Ker}\,\overline{\partial}_b$.
- If X is strongly pseudoconvex at some point of X and $\overline{\partial}_b$ has L^2 closed range, then $\dim \mathrm{Ker}\,\overline{\partial}_b = +\infty$ (Boutet de Monvel-Sjöstrand, Hsiao-Marinescu).

CR manifolds with group action

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- Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n+1$, $n \geq 2$.
- Now, we assume that
  - X admits a d-dim'l locally free connected compact Lie group action G with Lie algebra $\mathfrak{g}$.
  - The Lie group action G preserves ω_0 and CR structure. That is, $g^*\omega_0 = \omega_0$ and $dg(T^{1,0}X) = T^{1,0}X$, for every $g \in G$, $g : X \to X$.
- For any $\xi \in \mathfrak{g}$, ξ_X : the vector field on X induced by ξ. Let $\underline{\mathfrak{g}} = \mathrm{Span}\,(\xi_X;\, \xi \in \mathfrak{g})$.

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- Goal: Study $H^0_b(X)^G$ the space of global G-invariant L^2 CR functions.

CR momentum map

Definition

The momentum map associated to the form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{g}$, we have

$$\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)). \tag{1}$$

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- We will work under the following natural assumption.

Assumption

0 is a regular value of μ, the action of G on $\mu^{-1}(0)$ is free and the Levi form of X is positive definite near $\mu^{-1}(0)$.

- X is not necessarily strongly pseudoconvex.

CR reduction

- Let $Y := \mu^{-1}(0)$, $Y_G := \mu^{-1}(0)/G$.

Theorem (H/Huang, 2017)

Y_G is a strongly pseudoconvex CR manifold of dimension $2n - 2d + 1$ with natural CR structure $T^{1,0}Y_G$ induced from $T^{1,0}X$.

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- Is $H^0_b(X)^G \cong H^0_b(Y_G)$?

G-invariant Szegő projection

- Fix a G-invariant smooth Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$.
- Let $(\,\cdot\,|\,\cdot\,)$ and $(\,\cdot\,|\,\cdot\,)_{Y_G}$ be the L^2 inner products on $L^2(X)$ and $L^2(Y_G)$ induced by $\langle \cdot \mid \cdot \rangle$ respectively.
- Let $\overline{\partial}_b : \mathscr{C}^\infty(X) \to \Omega^{0,1}(X)$ and $\overline{\partial}_{b,Y_G} : \mathscr{C}^\infty(Y_G) \to \Omega^{0,1}(Y_G)$ be the tangential Cauchy-Riemann operators on X and Y_G respectively.
- We extend $\overline{\partial}_b$ and $\overline{\partial}_{b,Y_G}$ to L^2 spaces in the standard way.

$$\overline{\partial}_b : \mathrm{Dom}\,\overline{\partial}_b \subset L^2(X) \to L^2_{(0,1)}(X),$$
$$\overline{\partial}_{b,Y_G} : \mathrm{Dom}\,\overline{\partial}_{b,Y_G} \subset L^2(Y_G) \to L^2_{(0,1)}(Y_G).$$

G-invariant Szegő projection

- Put

$$
\begin{aligned}
H_b^0(X) &:= \left\{u \in L^2(X);\, \overline{\partial}_b u = 0\right\}, \\
H_b^0(X)^G &:= \left\{u \in H_b^0(X);\, h^* u = u, \quad \text{for every } h \in G\right\}, \\
H_b^0(Y_G) &:= \left\{u \in L^2(Y_G);\, \overline{\partial}_{b,Y_G} u = 0\right\}.
\end{aligned}
$$

- The G-invariant Szegő projection is the orthogonal projection

$$
S_G : L^2(X) \to H_b^0(X)^G
$$

  with respect to $(\cdot\,|\,\cdot)$.
- Let $S_{Y_G} : L^2(Y_G) \to H_b^0(Y_G)$ be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_{Y_G}$ (Szegő projection on Y_G).

The canonical Fredholm operator $\widehat{\sigma}$

- Let $\iota_G : \mathscr{C}^\infty(X)^G \to \mathscr{C}^\infty(Y_G)$ be the natural restriction,
- $\mathscr{C}^\infty(X)^G$: the space of G-invariant smooth functions on X.
- Let

$$\begin{aligned}\widehat{\sigma} : H^0_b(X)^G \cap \mathscr{C}^\infty(X)^G &\to H^0_b(Y_G), \\ u &\mapsto S_{Y_G} \circ E \circ \iota_G \circ f \circ u,\end{aligned} \tag{2}$$

  - $E : \mathscr{C}^\infty(Y_G) \to \mathscr{C}^\infty(Y_G)$: any elliptic pseudodifferential operator with principal symbol $p_E(x,\xi) = |\xi|^{-\frac{d}{4}}$,
  - $f \in \mathscr{C}^\infty(X)^G$: a specific G-invariant smooth function.

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- We can show that there exists a $C > 0$ such that $(\,\widehat{\sigma}u\,|\,\widehat{\sigma}u\,)_{Y_G} \leq C\,\|u\|^2$, for every $u \in H^0_b(X)^G \cap \mathscr{C}^\infty(X)$.
- Hence, we can extend $\widehat{\sigma}$ to $\widehat{\sigma} : H^0_b(X)^G \to H^0_b(Y_G)$.

Theorem I (H/Ma/Marinescu, 2019)

Suppose that $\overline{\partial}_{b,Y_G}$ has L^2 closed range and the Levi form is positive definite near $\mu^{-1}(0)$. The map

$$\widehat{\sigma} : H^0_b(X)^G \to H^0_b(Y_G)$$

is Fredholm. That is, $\mathrm{Ker}\,\widehat{\sigma}$ *and* $(\mathrm{Im}\,\widehat{\sigma})^\perp$ *are finite dimensional subspaces of the spaces* $\mathscr{C}^\infty(X) \cap H^0_b(X)^G$ *and* $\mathscr{C}^\infty(Y_G) \cap H^0_b(Y_G)$ *respectively.*

- In this work, we do not assume that $\overline{\partial}_b$ has closed range on X.
- The definition of $\widehat{\sigma}$ depends on the choice of elliptic pseudodifferential operator E.
- Up to lower order terms of E, the map $\widehat{\sigma}$ is a canonical choice.
- $\overline{\partial}_b$ is not hypoelliptic and not transversally elliptic in general.
- Theorem I establishes "quantization commutes with reduction" for some non-hypoelliptic operators.

Applications: Complex manifolds

- (L, h^L) : a holomorphic line bundle over a connected compact complex manifold (M, J),
- h^L is a Hermitian fiber metric of L.
- R^L : the curvature of (L, h^L).
- G : a connected compact Lie group with Lie algebra $\mathfrak{g}$. Assume that
    - G acts holomorphically on (M, J),
    - the action lifts to a holomorphic action on L,
    - h^L is preserved by the G-action.
    - $\omega := \frac{i}{2\pi} R^L$ is a G-invariant form.

Applications: Complex manifolds

- $\tilde{\mu} : M \to \mathfrak{g}^*$: the momentum map induced by ω. Assume that
    - $0 \in \mathfrak{g}^*$ is regular,
    - the action of G on $\tilde{\mu}^{-1}(0)$ is free.
- $M_0 := \tilde{\mu}^{-1}(0)/G$: a complex manifold with natural complex structure induced by J.
- $L_0 := L/G$: a holomorphic line bundle over M_0.

Theorem (Guillemin-Sternberg (1982))

Suppose that $R^L > 0$ on X. We have
$\dim H^0(M, L^m)^G = \dim H^0(M_0, L_0^m)$*, for every* $m \in \mathbb{N}^*$.

- $H^0(M_0, L_0^m)$: *the space of holomorphic sections on* M_0 *with values in* L_0^m,
- $H^0(M, L^m)^G$: *the space of* G*-invariant holomorphic sections with values in* L^m.

Remark

- Guillemin and Sternberg conjectured: "quantization commutes with reduction" holds for compact symplectic manifolds with compact connected Lie group G.
- When G is abelian, this conjecture was proved by Meinrenken (1996) and Vergne (1996).
- The remaining nonabelian case was first proved by Meinrenken (1998) using the symplectic cut techniques of Lerman, and then by Tian and Zhang (1998) using analytic localization techniques by Bismut-Lebeau.
- Paradan developed a K-theoretic approach for Guillemin and Sternberg conjecture.

Remark

- When symplectic manifold is non-compact and momentum map is proper, "quantization commutes with reduction problem (Vergne's conjecture in her ICM 2006 plenary lecture)," was solved by Ma-Zhang.
- Paradan gave a new proof for Vergne's conjecture.
- When manifold and group are both non-compact: many works, Mathai, Zhang, Hochs, etc.

Applications: Complex manifolds

- Let X be the circle bundle of (L^*, h^{L^*}), i.e. $X := \left\{ v \in L^*;\ |v|^2_{h^{L^*}} = 1 \right\}$.
- X is a compact strongly pseudoconvex CR manifold with a group action G.
- X admits a S^1 action $e^{i\theta}$: $e^{i\theta} \circ (z, \lambda) := (z, e^{i\theta}\lambda)$, where λ denotes the fiber coordinate of X.
- $\overline{\partial}_b$ is not hypoelliptic but transversally elliptic with respect to the S^1 action.

Applications: Complex manifolds

- For every $m \in \mathbb{Z}$, let

$$H^0_{b,m}(X)^G := \left\{u \in H^0_b(X)^G;\ (e^{i\theta})^* u = e^{im\theta} u, \text{for every } e^{i\theta} \in S^1\right\},$$
$$H^0_{b,m}(Y_G) := \left\{u \in H^0_b(Y_G);\ (e^{i\theta})^* u = e^{im\theta} u, \text{for every } e^{i\theta} \in S^1\right\}.$$

- We have
  - $H^0_b(X)^G := \oplus_{m\in\mathbb{Z}} H^0_{b,m}(X)^G$, $H^0_b(Y_G) := \oplus_{m\in\mathbb{Z}} H^0_{b,m}(Y_G)$,
  - $\widehat{\sigma} : H^0_{b,m}(X)^G \to H^0_{b,m}(Y_G)$, for every $m \in \mathbb{Z}$.
- For every $m \in \mathbb{Z}$,

$$H^0(M, L^m)^G \cong H^0_{b,m}(X)^G, \quad H^0(M_0, L^m_0) \cong H^0_{b,m}(Y_G).$$

- From Theorem I, we deduce that if $|m| \gg 1$, then $\dim H^0_{b,m}(X)^G = \dim H^0_{b,m}(Y_G)$.

Applications: Complex manifolds

- Ma-Zhang showed that the map

$$\begin{aligned}\rho : H^0(M, L^m)^G &\to H^0(M_0, L_0^m), \\ u &\mapsto m^{-\frac{d}{4}} B_{M_0,m} \circ \iota_G \circ f \circ u,\end{aligned} \tag{3}$$

  is an isomorphism if m is large enough,
- $B_{M_0,m} : L^2(M_0, L_0^m) \to H^0(M_0, L_0^m)$: the orthogonal projection (Bergman projection).
- When we change $m^{-\frac{d}{4}}$ in (3) to any m-dependent function with order $m^{-\frac{d}{4}} + O(m^{-\frac{4}{d}-1})$, we still have an isomorphism between $H^0(M, L^m)^G$ and $H^0(M_0, L_0^m)$ for m large.

Applications: Complex manifolds

- In this work, we only assume the Levi form is positive definite near $\mu^{-1}(0)$.
- As an application of Theorem I, we deduce

Theorem

With the notations and assumptions above and suppose that R^L is positive near $\tilde{\mu}^{-1}(0)$. Then, for $|m|$ large, we have

$$\dim H^0(M, L^m)^G = \dim H^0(M_0, L_0^m).$$

Applications: Sasakian manifolds

- Let $(X, T^{1,0}X)$ be a compact strongly pseudoconvex CR manifold.
- We say that X is torsion free if there is a non-vanishing global real vector field $T \in \mathscr{C}^\infty(X, TX)$ such that
  - T preserves the CR structure $T^{1,0}X$,
  - T, $T^{1,0}X \oplus T^{0,1}X$ generate the complex tangent bundle of X.
- We call T CR Reeb vector field on X.
- Ornea and Verbitsky: A $(2n+1)$-dimensional smooth manifold X is a Sasakian manifold if and only if X is a torsion free strongly pseudoconvex CR manifold.

Applications: Sasakian manifolds

- X is a quasi-regular (regular) Sasakian manifold if the flow of T induces a locally free (free) S^1-action on X.
- X is an irregular Sasakian manifold if there is an orbit of the flow of T which is non-compact.
- In this case, the flow of T induces a transversal CR $\mathbb{R}$-action on X.
- We now assume that X is an irregular Sasakian manifold with a CR Reeb vector field T and suppose that the Lie group G preserves T and CR structure on X.

- Consider the operators

$$
\begin{aligned}
& -i\mathscr{L}_T : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X), \\
& -i\mathscr{L}_{\widehat{T}} : \mathscr{C}^\infty(Y_G) \to \mathscr{C}^\infty(Y_G),
\end{aligned}
$$

  - $\widehat{T}$ is the CR Reeb vector field on Y_G,
  - $\mathscr{L}_T$, $\mathscr{L}_{\widehat{T}}$ denote the Lie derivative of T and $\widehat{T}$ respectively.
- We extend $-i\mathscr{L}_T$ and $-i\mathscr{L}_{\widehat{T}}$ to L^2 spaces by their weak maximal extension.

Theorem (H/Herrmann/Li, 2017)

We have that $\mathrm{Spec}\,(-i\mathscr{L}_T)$ *is countable and every element in* $\mathrm{Spec}\,(-i\mathscr{L}_T)$ *is an eigenvalue of* $-i\mathscr{L}_T$*, where* $\mathrm{Spec}\,(-i\mathscr{L}_T)$ *denotes the spectrum of* $-i\mathscr{L}_T$*.*

- Put

$$\mathrm{Spec}\,(-i\mathscr{L}_T) = \{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{R},$$
$$\mathrm{Spec}\,(-i\mathscr{L}_{\widehat{T}}) = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{R},$$
$$H^0_{b,\alpha}(X)^G := \left\{u \in H^0_b(X)^G;\ -i\mathscr{L}_T u = \alpha u\right\},\quad \alpha \in \mathrm{Spec}\,(-i\mathscr{L}_T),$$
$$H^0_{b,\beta}(Y_G) := \left\{v \in H^0_b(Y_G);\ -i\mathscr{L}_{\widehat{T}} v = \beta u\right\},\quad \beta \in \mathrm{Spec}\,(-i\mathscr{L}_{\widehat{T}}).$$

Applications: Sasakian manifolds

- $\overline{\partial}_b$ is transversally elliptic with respect to the $\mathbb{R}$ action.
- $H^0_{b,\alpha}(X)^G$ and $H^0_{b,\beta}(Y_G)$ are finite dimensional subspaces of $\mathscr{C}^\infty(X)^G$ and $\mathscr{C}^\infty(Y_G)$ respectively, for every $\alpha \in \mathrm{Spec}\,(-i\mathscr{L}_T)$, $\beta \in \mathrm{Spec}\,(-i\mathscr{L}_{\widehat{T}})$.
- $H^0_b(X)^G = \oplus_{\alpha \in \mathrm{Spec}\,(-i\mathscr{L}_T)} H^0_{b,\alpha}(X)^G$,
  $H^0_b(Y_G) = \oplus_{\beta \in \mathrm{Spec}\,(-i\mathscr{L}_{\widehat{T}})} H^0_{b,\beta}(Y_G)$.

Quantization commutes with reduction for irregular Sasakian manifolds

Theorem II (H/Ma/Marinescu, 2019)

There is a $N \in \mathbb{N}$ such that the map

$$\widehat{\sigma} : H^0_{b,\alpha_k}(X)^G \to H^0_{b,\alpha_k}(Y_G)$$

is an isomorphism, for every $k \geq N$ and if $\beta_k \neq \alpha_k$, where $k \geq N$, then $\dim H^0_{b,\beta_k}(Y_G) = 0$.

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is an isomorphism, for every $k \geq N$ *and if* $\beta_k \neq \alpha_k$, *where* $k \geq N$, *then* $\dim H^0_{b,\beta_k}(Y_G) = 0$.

- It was shown by Marinescu and Yeganefar that $\overline{\partial}_{b,Y_G}$ has L^2 closed range.
- In the rest of this talk, we will sketch the proof of Theorem I.

G-invariant Kohn Laplacian

- Consider $\overline{\partial}_{b,G} : \mathrm{Dom}\,\overline{\partial}_{b,G} \subset L^2(X)^G \to L^2_{(0,1)}(X)^G$,
  - $L^2_{(0,1)}(X)^G$: the space of G-invariant L^2 $(0,1)$ forms,
  - $\mathrm{Dom}\,\overline{\partial}_{b,G} = \left\{u \in L^2(X)^G; \overline{\partial}_b u \in L^2_{(0,1)}(X)^G\right\}$,
  - $\overline{\partial}_{b,G} u = \overline{\partial}_b u$, for every $u \in \mathrm{Dom}\,\overline{\partial}_{b,G}$.
- Let $\overline{\partial}^*_{b,G} : \mathrm{Dom}\,\overline{\partial}^*_{b,G} \subset L^2_{(0,1)}(X)^G \to L^2(X)^G$ be the Hilbert space adjoint of $\overline{\partial}_{b,G}$ with respect to $(\,\cdot\,|\,\cdot\,)$.

G-invariant Kohn Laplacian

- Let

$$\Box_{b,G} = \overline{\partial}^*_{b,G}\overline{\partial}_{b,G} : \mathrm{Dom}\,\Box_{b,G} \subset L^2(X)^G \to L^2(X)^G$$

denote the (Gaffney extension) of the G-invariant Kohn Laplacian given by

$$\mathrm{Dom}\,\Box_{b,G} = \left\{ s \in L^2(X)^G;\ s \in \mathrm{Dom}\,\overline{\partial}_{b,G},\ \ \overline{\partial}_{b,G}s \in \mathrm{Dom}\,\overline{\partial}^*_{b,G} \right\},$$

$\Box_{b,G}s = \overline{\partial}^*_{b,G}\overline{\partial}_{b,G}s$ for $s \in \mathrm{Dom}\,\Box_{b,G}$.

- $\Box_{b,G}$ is self-adjoint.

Closed range property for G-invariant Kohn Laplacian

- By using some kind of Kohn's hypoelliptic estimates, we can show that

Theorem (H/Ma/Marinescu, 2019)

Recall that we work with the assumption that the Levi form is positive near $\mu^{-1}(0)$. The operator $\Box_{b,G} : \mathrm{Dom}\,\Box_{b,G} \subset L^2(X)^G \to L^2(X)^G$ has closed range

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- The Kohn Laplacian $\Box_b : \mathrm{Dom}\,\Box_b \subset L^2(X) \to L^2(X)$ may not have closed range.

G-invariant Szegő kernel

- Let $S_G(x,y) \in \mathscr{D}'(X \times X)$ be the distribution kernel of the orthogonal projection $S_G : L^2(X) \to \operatorname{Ker} \Box_{b,G} = H^0_b(X)^G$.

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- We have

$$S_G(x,y) = \int_G S(x, h \circ y) d\mu(h) \text{ on } X \times X, \tag{4}$$

  - $d\mu = d\mu(h)$: the Haar measure on G with $\int_G d\mu(h) = 1$,
  - $S(x,y) \in \mathscr{D}'(X \times X)$ is the distribution kernel of the orthogonal projection $S : L^2(X) \to \operatorname{Ker} \Box_b$.

G-invariant Szegő kernel

- Let $S_G(x,y) \in \mathscr{D}'(X \times X)$ be the distribution kernel of the orthogonal projection $S_G : L^2(X) \to \mathrm{Ker}\,\Box_{b,G} = H^0_b(X)^G$.
- We have

$$S_G(x,y) = \int_G S(x, h \circ y) d\mu(h) \text{ on } X \times X, \tag{4}$$

  - $d\mu = d\mu(h)$: the Haar measure on G with $\int_G d\mu(h) = 1$,
  - $S(x,y) \in \mathscr{D}'(X \times X)$ is the distribution kernel of the orthogonal projection $S : L^2(X) \to \mathrm{Ker}\,\Box_b$.
- When X is strongly pseudoconvex and $\Box_b$ has closed range, we can study $S_G(x,y)$ by using (4) and Boutet de Monvel and Sjöstrand's classical result for the Szegő kernel $S(x,y)$.

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- Since the Kohn Laplacian $\Box_b$ may not have closed range, it is difficult to study $S_G(x,y)$ by using (4).

Spectral kernel for Kohn Laplacian

- $\Box_b$ is self-adjoint and the spectrum of $\Box_b$ is contained in $\overline{\mathbb{R}}_+$.
- For $\lambda \geq 0$, set $H^0_{b,\leq\lambda}(X) := \mathrm{Ran}\, E\big([0,\lambda]\big) \subset L^2(X)$,
  - $E([0,\lambda]))$:the spectral projection of $\Box_b$,
  - E: the spectral measure of $\Box_b$.
- Let

$$S_{\leq\lambda} : L^2(X) \to H^0_{b,\leq\lambda}(X)$$

  be the orthogonal projection with respect to the product $(\,\cdot\,|\,\cdot\,)$.
- Let $S_{\leq\lambda}(x,y) \in \mathscr{D}'(X \times X)$ be the distribution kernels of $S_{\leq\lambda}$.

Theorem (H/Marinescu, 2017)

Assume that the Levi form is positive on an open set $D \Subset X$. Then for every $\lambda > 0$, $S_{\leq\lambda}(x,y)$ is a complex Fourier integral operator on D of the form

$$\begin{aligned} & S_{\leq\lambda}(x,y) \equiv \int_0^{\infty} e^{i\varphi(x,y)t} s(x,y,t)dt, \\ & s(x,y,t) \sim \sum_{j=0}^{+\infty} t^{n-j} s_j(x,y), \\ & s_0(x,y) \neq 0, \\ & \operatorname{Im}\varphi \geq 0, \quad \varphi(x,x) = 0. \end{aligned} \tag{5}$$

- From the closed range property for G-invariant Kohn Laplacian, we can show that

Theorem

There is a $\lambda_0 > 0$ *such that*

$$S_G(x,y) = \int_G S_{\leq \lambda_0}(x, h \circ y) d\mu(h) \ \ \textit{on } X \times X. \tag{6}$$

G-invariant Szegő kernel asymptotics

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Theorem

There is a $\lambda_0 > 0$ such that

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- From (5), (6) and note that the Levi form is positive near $\mu^{-1}(0)$, we can study $S_G(x,y)$ near $\mu^{-1}(0)$.
- By using Kohn's estimates, we can show that S_G is smoothing away $\mu^{-1}(0)$.

G-invariant Szegő kernel asymptotics

Theorem III (H/Ma/Marinescu, 2019)

- *S_G is smoothing outside $\mu^{-1}(0)$.*
- *In an open set U of $\mu^{-1}(0)$, we have*

$$S_G(x,y) \equiv \int_0^\infty e^{i\Phi(x,y)t} a(x,y,t)dt \quad \text{on } U \times U,$$

  - $a(x,y,t) \sim \sum_{j=0}^{\infty} a_j(x,y) t^{n-\frac{d}{2}-j}$ in $S^{n-\frac{d}{2}}_{1,0}(U \times U \times \mathbb{R}_+)$,
  - $d_x\Phi(x,x) = -d_y\Phi(x,x) = -\omega_0(x)$, $\forall x \in \mu^{-1}(0)$,
  - $\operatorname{Im}\Phi(x,y) \geq 0$, $\operatorname{Im}\Phi(x,x) \approx d(x,\mu^{-1}(0))^2$.

The outline of the proof of Theorem I

- Consider $\widehat{\sigma}$ as a map acting on $L^2(X)$:

$$\widehat{\sigma} : L^2(X) \to H^0_b(Y_G) \subset L^2(Y_G), \quad u \to S_{Y_G} \circ E \circ \iota_G \circ f \circ S_G \circ u.$$

- Let $\widehat{\sigma}^* : L^2(Y_G) \to \mathscr{D}'(X)$ be the adjoint of $\widehat{\sigma}$.
- From Theorem III and by developing some kind of complex Fourier integral operators calculation, we can show that
    - $F := \widehat{\sigma}^* \widehat{\sigma} : L^2(X) \to \mathscr{D}'(X)$ is the same type of operator as S_G,
    - $F = C_0(I - R)S_G$, C_0 is a constant, R is also the same type of operator as S_G.
    - We take E so that the order of F is the same as the order of S_G.

The outline of the proof of Theorem I

- We can take f so that
  - the leading symbol of R vanishes at $\mathrm{diag}\,(\mu^{-1}(0)\times\mu^{-1}(0))$,
  - $R: H^s(X)\to H^{s+\varepsilon}(X)$ is continuous, for every $s\in\mathbb{Z}$, where $\varepsilon>0$ is a constant and $H^s(X)$ denotes the Sobolev space of order s on X.
- From $F=\widehat{\sigma}^*\widehat{\sigma}=C_0(I-R)$ on $H^0_b(X)^G$ and the regularity property of R, we can show that
  - the kernel of $F: H^0_b(X)^G\to H^0_b(X)^G$ is a finite dimensional subspace of $\mathscr{C}^\infty(X)\cap H^0_b(X)^G$.
- Since $\mathrm{Ker}\,\widehat{\sigma}\subset\mathrm{Ker}\,F$, $\mathrm{Ker}\,\widehat{\sigma}$ is a finite dimensional subspace of $\mathscr{C}^\infty(X)\cap H^0_b(X)^G$.

The outline of the proof of Theorem I

- Similarly, we can repeat the argument above with minor change and deduce that
  - the kernel of the map

$$\hat{F} = \widehat{\sigma}\widehat{\sigma}^* : H^0_b(Y_G) \rightarrow H^0_b(Y_G)$$

    is a finite dimensional subspace of $\mathscr{C}^\infty(Y_G) \cap H^0_b(Y_G)$.

- Since $(\mathrm{Im}\,\widehat{\sigma})^\perp \subset \mathrm{Ker}\,\hat{F}$, $(\mathrm{Im}\,\widehat{\sigma})^\perp$ is a finite dimensional subspace of $\mathscr{C}^\infty(Y_G) \cap H^0_b(Y_G)$.

The outline of the proof of Theorem I

- Similarly, we can repeat the argument above with minor change and deduce that
    - the kernel of the map

$$\hat{F} = \widehat{\sigma}\widehat{\sigma}^* : H_b^0(Y_G) \to H_b^0(Y_G)$$

   is a finite dimensional subspace of $\mathscr{C}^\infty(Y_G) \cap H_b^0(Y_G)$.
- Since $(\mathrm{Im}\,\widehat{\sigma})^\perp \subset \mathrm{Ker}\,\hat{F}$, $(\mathrm{Im}\,\widehat{\sigma})^\perp$ is a finite dimensional subspace of $\mathscr{C}^\infty(Y_G) \cap H_b^0(Y_G)$.
- We can also apply the method of the proof to study some extension problems in several complex variables.

Microlocal isometric map σ

- By using spectral theory, we will show that there is a self-adjoint bounded operator
  - $\sqrt{N_+} : L^2(X) \to L^2(X)$,
  - $\sqrt{N_+} S_G = S_G \sqrt{N_+}$ on $L^2(X)$,
  - $\sqrt{N_+}(I - R)\sqrt{N_+} = I - P$ on $L^2(X)$, where P is the orthogonal projection from $L^2(X)$ onto $\mathrm{Ker}\,(I - R)$.
- Let $\sigma := \frac{1}{\sqrt{C_0}} \widehat{\sigma} \circ S_G \circ \sqrt{N_+} : H^0_b(X)^G \to H^0_b(Y_G)$.
- $\sigma^* \sigma = \frac{1}{C_0} \sqrt{N_+} \widehat{\sigma}^* \widehat{\sigma} \sqrt{N_+} = \sqrt{N_+}(I - R)\sqrt{N_+} = I - P$.

Microlocal isometric map σ

- We have

$$(\,\sigma u\,|\,\sigma v\,)_{Y_G} = (\,\sqrt{N_+}(I-R)\sqrt{N_+}u\,|\,v\,) = (\,u\,|\,v\,) - (\,P\,u\,|\,v\,),$$

  for every $u, v \in H^0_b(X)^G$.
- Note that P is smoothing and for $u \in H^0_b(X)^G$, $Pu = 0$ if and only of $u \in (\mathrm{Ker}\,\sigma)^\perp$.
- σ is microlocally isometric.

Theorem (H/Ma/Marinescu, 2019)

Suppose that $\overline{\partial}_{b,Y_G}$ has L^2 closed range and the Levi form is positive near $\mu^{-1}(0)$. The map

$$\sigma : H^0_b(X)^G \to H^0_b(Y_G)$$

is Fredholm and

$$(\, \sigma u \,|\, \sigma v \,)_{Y_G} = (\, u \,|\, v \,), \quad u, v \in (\mathrm{Ker}\, \sigma)^{\perp}.$$