Geometric quantization of the reduced space of a product of coadjoint orbits

Joint work with Jia Ji

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0. The triple reduced product Space: Motivation

- Let $G = SU(3)$, and let T be the maximal torus. Let the Lie algebras be denoted $\mathbf{g}$ and $\mathbf{t}$ respectively.

  We are considering the symplectic quotient of the product of three coadjoint orbits of $SU(3)$

  $$M = \mathcal{O}_{\xi_1} \times \mathcal{O}_{\xi_2} \times \mathcal{O}_{\xi_3} // G$$

  where $\xi_j \in \mathbf{t}$ are in the Lie algebra of the maximal torus of $SU(3)$ (in other words they are diagonal matrices with purely imaginary entries).

- The moment map for each (co)adjoint orbit is the inclusion map into the Lie algebra $\mathbf{g}$.

- So if $X, Y, Z \in \mathcal{O}_\lambda \times \mathcal{O}_\mu \times \mathcal{O}_\nu$, the moment map is $\phi(X, Y, Z) = X + Y + Z$.

- This space has dimension 2 (because the dimension of each of the orbits is 6 and the moment map condition reduces the dimension by 8 , while the quotient by the group action reduces by a further 8: $18 - 8 - 8 = 2$)

- The triple reduced product space may be identified with a polygon space, a space of triangles in $\mathfrak{su}(3)$ with vertices in specific coadjoint orbits.
- These spaces are a prototype for flat connections on the three-punctured sphere, with the holonomy around each puncture constrained to lie in a prescribed conjugacy class. (See LJ, *Math. Ann.* 1994.)

- The orbit method (Kirillov) has many applications in geometry.
- A tuple of matrices may be identified with a Higgs field.
- In the paper "The triple reduced product and Hamiltonian flows" (L. Jeffrey, S. Rayan, G. Seal, P. Selick, J. Weitsman, in XXXV WGMP Proceedings), the main objective was to identify a Hamiltonian function which was the moment map for a circle action. We were able to do this only indirectly, by choosing an auxiliary function which maps the triple reduced product onto the unit interval, and defining the moment map indirectly as a definite integral involving the auxiliary function.
- Symplectic volume of triple reduced product is known (Suzuki-Takakura [ST08]; LJ- Jia Ji, arXiv:1804.06474)

- Assuming that 0 is a regular value of the moment map, the triple reduced product is homeomorphic to S^2.

- Our dimension count tells us that it has dimension 2. For generic values of the moment map, the zero level set of the moment map is a manifold, and the G action is free so the quotient is also a manifold. Since the Kirwan map is surjective, $\dim H^0 \leq 1$ (because the space is connected), $\dim H^1 = 0$ –because the orbits have no odd-dimensional cohomology – and $\dim H^2 \leq 1$ (by Poincaré duality). Hence the reduced space must be either empty or S^2.

1. Background

Let $\xi_1, \ldots, \xi_N \in \mathbf{t}$.

Assumption 1: All of $\mathcal{O}(\xi_1), \cdots, \mathcal{O}(\xi_N)$ are diffeomorphic to the homogeneous space G/T. This assumption is equivalent to the assumption that all of the stabilizer groups $\mathrm{Stab_G}(\xi_1), \cdots, \mathrm{Stab_G}(\xi_N)$ are conjugate to the chosen maximal torus T. If all of $\xi_1, \cdots, \xi_N$ are contained in $\mathfrak{t}^* \subseteq \mathfrak{g}^*$, then this assumption is saying that

$$\mathrm{Stab_G}(\xi_1) = \cdots = \mathrm{Stab_G}(\xi_N) = \mathrm{T}.$$

Remark: Since every coadjoint orbit $\mathcal{O}(\xi)$ can be written as $\mathcal{O}(\xi')$ for some $\xi' \in \mathfrak{t}^* \subseteq \mathfrak{g}^*$, we can always assume that $\underline{\xi} = (\xi_1, \cdots, \xi_N)$ satisfies that $\xi_j \in \mathfrak{t}^* \subseteq \mathfrak{g}^*$ for all j.

The Cartesian product $\mathcal{M}(\underline{\xi}) = \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$ carries a natural symplectic structure $\omega_{\underline{\xi}}$ defined by:

$$\omega_{\underline{\xi}} := \pi_1^* \omega_{\mathcal{O}(\xi_1)} + \cdots + \pi_N^* \omega_{\mathcal{O}(\xi_N)} \tag{1}$$

where $\pi_j : \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N) \to \mathcal{O}(\xi_j)$ is the projection onto the j-th component.

Let G act on $\mathcal{M}(\underline{\xi}) = \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$ by the diagonal action Δ:

$$\Delta(g)(\eta_1, \cdots, \eta_N) := (K(g)(\eta_1), \cdots, K(g)(\eta_N)) \tag{2}$$

for all $g \in G$, $\eta_j \in \mathcal{O}(\xi_j)$. Here $K(g)$ denotes the (co)adjoint action of g.

The symplectic form $\omega_{\underline{\xi}}$ is clearly G-invariant, and we also have the following.

Proposition: The diagonal action Δ of G on $(\mathcal{M}(\underline{\xi}), \omega_{\underline{\xi}})$ is a Hamiltonian G-action with the moment map $\mu_{\underline{\xi}} : \mathcal{M}(\underline{\xi}) \to \mathfrak{g}^*$ being:

$$\mu_{\underline{\xi}}(\underline{\eta}) = \sum_{j=1}^{N} \eta_j \tag{3}$$

for all $\underline{\eta} := (\eta_1, \cdots, \eta_N) \in \mathcal{M}(\underline{\xi})$.

We assume that:

Assumption 2: $0 \in \mathfrak{g}^*$ is a regular value for $\mu_{\underline{\xi}} : \mathcal{M}(\underline{\xi}) \to \mathfrak{g}^*$ and $\mu_{\underline{\xi}}^{-1}(0) \neq \emptyset$.

Remark: By Sard's theorem, the set where the previous two assumptions hold is nonempty and has nonempty interior in $\mathfrak{t}^* \times \cdots \times \mathfrak{t}^*$.

Then, the level set $\mathcal{M}_0(\underline{\xi}) := \mu_{\underline{\xi}}^{-1}(0)$ is a closed, thus compact, submanifold of $\mathcal{M}(\underline{\xi})$ and the diagonal action Δ of G restricts to an action on $\mathcal{M}_0(\underline{\xi})$. Therefore, we can form the quotient space with respect to this action of G on $\mathcal{M}_0(\underline{\xi})$:

$$\mathcal{M}(\underline{\xi}) := \mathcal{M}_0(\underline{\xi})/G. \tag{4}$$

The above quotient space will also be denoted by $M//G$. Note that this quotient space is compact.

If the G-action on $\mathcal{M}_0(\underline{\xi})$ is free and proper (in our situation, properness is automatically satisfied), then the quotient space $\mathcal{M}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ is a smooth manifold. However, in our situation, the G-action on $\mathcal{M}_0(\underline{\xi})$ is in general not free. Hence, in general the quotient space is only an orbifold. To avoid this complication, we will assume:

Assumption 3: The quotient space $\mathcal{M}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ is a smooth compact manifold.

Remark: The above assumption will put further restrictions on which $\underline{\xi} \in \mathfrak{t}^* \times \cdots \times \mathfrak{t}^*$ we can choose as initial data. Thus we only choose initial data from the following set in this talk:

$$\mathcal{A}' := \left\{ \underline{\xi} \in \overbrace{\mathfrak{t}^* \times \cdots \times \mathfrak{t}^*}^{N} \ : \ \text{previous 3 assumptions hold} \right\} \tag{5}$$

Suzuki and Takakura also made this assumption in their paper [ST08] (in Section 2.3). It seems reasonable to us to assume that even after Assumption 3 is imposed, the initial data set $\mathcal{A}'$ is still nonempty and still has nonempty interior in $\mathfrak{t}^* \times \cdots \times \mathfrak{t}^*$. Notice that since the elements in the center of G always act trivially on $\mathcal{M}(\underline{\xi})$ and $\mathcal{M}_0(\underline{\xi})$, Assumption 3 is valid if $PG = G/Z(G)$ acts freely on $\mathcal{M}_0(\underline{\xi})$. This happens for $G = SU(n)$ if all the coadjoint orbits $\mathcal{O}(\xi_i)$ are generic.

Then, we have the following well known theorem:

Theorem:[Marsden-Weinstein] The smooth compact manifold $\mathcal{M}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ carries a unique symplectic structure $\omega(\underline{\xi})$ such that

$$i^*\omega_{\underline{\xi}} = \pi^*\omega(\underline{\xi}) \tag{6}$$

where $i : \mathcal{M}_0(\underline{\xi}) \hookrightarrow \mathcal{M}(\underline{\xi})$ is the inclusion map and $\pi : \mathcal{M}_0(\underline{\xi}) \to \mathcal{M}(\underline{\xi})$ is the associated projection map.

2. The case $G = SU(3)$, $N = 3$

In this section, we study 3-fold reduced products, or triple reduced products for $G = SU(3)$. See [TRP1], [TRP2] for recent studies about these objects. Our focus is on the symplectic volume of triple reduced products.

The Setup for the case $G = SU(3)$

- Let $G = SU(3)$ and let T be its standard maximal torus, i.e., T consists of diagonal matrices in $SU(3)$.
- In this case, we know that the corresponding Weyl group W is isomorphic to the permutation group $\mathfrak{S}_3$.
- The Weyl group W acts on $\mathfrak{t}^* \cong \mathfrak{t}$ by permutations of diagonal entries.
- The elements

$$H_1 := 2\pi i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 := 2\pi i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

  in $\mathfrak{t}$ are generators of the integral lattice $\exp^{-1}(I) \subset \mathfrak{t}$. The elements H_1, H_2 form a basis of $\mathfrak{t}$.

- Let ω_1, ω_2 be the basis of $\mathfrak{t}^*$ dual to H_1, H_2, i.e., $\omega_i(H_j) = \delta_{ij}$. Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, ω_1, ω_2 correspond to the elements

$$\Omega_1 := \frac{2\pi i}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Omega_2 := \frac{2\pi i}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

  in $\mathfrak{t}$, respectively.

- Let $\mathfrak{t}^*_+ := \mathbb{R}_{\geq 0}\omega_1 + \mathbb{R}_{\geq 0}\omega_2$ and $\Lambda_+ := \mathbb{Z}_{\geq 0}\omega_1 + \mathbb{Z}_{\geq 0}\omega_2$. So $\mathfrak{t}^*_+$ is a positive Weyl chamber and Λ_+ is the associated set of dominant integral weights. Any element ξ of $\mathfrak{t}^*_+$ or Λ_+ can be written as

$$\xi = (\ell - m)\omega_1 + m\omega_2, \quad \ell \geq m \geq 0. \tag{7}$$

- Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, ξ corresponds to the element

$$X = (\ell - m)\Omega_1 + m\Omega_2. \tag{8}$$

- Every coadjoint orbit can be written as $\mathcal{O}_\xi$ for some $\xi \in FWC$, and in this case, $\mathcal{O}_\xi \cap \mathfrak{t}^*$ is the W-orbit through ξ, and $\mathcal{O}_\xi \cap FWC = \{\xi\}$.

- If $\xi = (\ell - m)\omega_1 + m\omega_2 \in FWC$ with $\ell > m > 0$, then $\mathrm{Stab}_{\mathrm{G}}(\xi) = \mathrm{T}$ and $\mathcal{O}_\xi$

is diffeomorphic to the homogeneous space G/T.

- Let $\xi_1, \xi_2, \xi_3 \in FWC$ so that $\xi_i = (\ell_i - m_i)\omega_1 + m_i\omega_2$ with $\ell_i > m_i > 0$. Let $\underline{\xi} := (\xi_1, \xi_2, \xi_3)$.
- Then $\underline{\xi}$ determines a triple reduced product $(\mathcal{M}(\underline{\xi}), \omega(\underline{\xi}))$.

3. Symplectic Volume of a Triple Reduced Product

- By nonabelian localization [JK95], $\int_{\mathcal{M}} e^{i\omega}$ can be expressed as a finite sum of contributions indexed by the fixed point set $\mathcal{M}^T$ of $\mathcal{M}$ under the action of the maximal torus T:

$$\mathcal{M}^T = \{(w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3) \,:\, w_1, w_2, w_3 \in W\}\,. \tag{9}$$

- More precisely, we have

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{\underline{w} \in W^3} \int_{X \in \mathbf{t}} \varpi^2(X) \frac{e^{i\langle \mu(\underline{w} \cdot \underline{\xi}), X \rangle}}{\mathrm{e}_{\underline{w} \cdot \underline{\xi}}(X)}\, dX, \tag{10}$$

- Here

- $\underline{w} = (w_1, w_2, w_3) \in W^3, \underline{\xi} = (\xi_1, \xi_2, \xi_3)$

$$\underline{w} \cdot \underline{\xi} := w_1 \cdot \xi_1 + w_2 \cdot \xi_2 + w_3 \cdot \xi_3 \tag{11}$$

- $\varpi(X) = \prod_\alpha \langle \alpha, X \rangle$ with α running over all positive roots of $G = SU(3)$

- $\mathrm{e}_F(X)$ is the equivariant Euler class of the normal bundle to the fixed point F. In this case,

$$\mathrm{e}_{\underline{w} \cdot \underline{\xi}}(X) = sgn(\underline{w}) \varpi^3(X), \tag{12}$$

where $\mathrm{s}gn(\underline{w}) := \mathrm{s}gn(w_1)\mathrm{s}gn(w_2)\mathrm{s}gn(w_3)$.

- This is the Fourier transform of the Duistermaat-Heckman oscillatory integral evaluated at 0.

- The DH oscillatory integral decomposes as a sum of finitely many terms. None of these terms separately admits a Fourier transform, but it is possible to define a Fourier transform of each term provided one picks polarizations consistently at each term (see [GLS96]).
- In the special case when $\mathbf{t} = \mathbf{R}$, a choice of a polarization is a choice to replace $\mathbf{R}$ by $\mathbf{R} + \mathbf{i}\epsilon$ where the choice of polarization is the choice of sign of ϵ.

Theorem:

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{\underline{w}\in W^3} sgn(\underline{w}) \int_{X\in\mathbf{t}} \frac{e^{i\langle \mu(\underline{w}\cdot\underline{\xi}),X\rangle}}{\varpi(X)}\, dX. \tag{13}$$

The symplectic volume of the reduced space $\mu_{\eta,T}^{-1}(0)/T$ of the Hamiltonian system $(\mathcal{O}_\eta, \omega_\eta, T, \mu_{\eta,T})$, where $\mu_{\eta,T} : \mathcal{O}_\eta \hookrightarrow \mathbf{t}^* \subset \mathbf{g}^*$ is the moment map associated to the Hamiltonian group action (in this case, the coadjoint action) on $\mathcal{O}_\eta$ by the standard maximal torus T, is expressed by the following formula, known from [GLS] and [JK95] (using Atiyah-Bott-Berline-Vergne localization).

- **Theorem:**

$$\mathrm{S}Vol(\mu_{\eta,T}^{-1}(0)/T) = \frac{1}{2\pi i}\sum_{w\in W}\mathrm{sgn}(w)\int_{X\in\mathbf{t}}\frac{e^{i\langle w\cdot\eta,X\rangle}}{\varpi(X)}\,dX. \tag{14}$$

- Let

$$f(\eta) := 2\pi i\,\mathrm{Vol}(\mu_{\eta,T}^{-1}(0)/T) = \sum_{w\in W}\mathrm{sgn}(w)\int_{X\in\mathbf{t}}\frac{e^{i\langle w\cdot\eta,X\rangle}}{\varpi(X)}\,dX. \tag{15}$$

- Then, by writing $w_2 = w_1 w_1^{-1} w_2,\ w_3 = w_1 w_1^{-1} w_3$ and letting $w_2' = w_1^{-1} w_2,\ w_3' = w_1^{-1} w_3$, we obtain

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{w_2'\in W}\sum_{w_3'\in W}\mathrm{sgn}(w_2')\mathrm{sgn}(w_3') f(\xi_1 + w_2'\cdot\xi_2 + w_3'\cdot\xi_3). \tag{16}$$

- On the other hand, it is known from [JK95] (from Atiyah-Bott-Berline-Vergne localization formula and nonabelian localization) that

Theorem:

$$\mathrm{Vol}(\mu_{\eta,T}^{-1}(0)/T) = \sum_{w\in W} sgn(w) H_{\underline{\beta}}(w\cdot\eta) \tag{17}$$

- Here $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ and $\beta_1, \beta_2, \beta_3$ are the positive roots of $SU(3)$, and

$$H_{\underline{\beta}}(\xi) := \mathrm{vol}_a \left\{ (s_1, s_2, s_3) \in \mathbb{R}^3_{\geq 0} : \sum_{i=1}^{3} s_i\beta_i = \xi \right\} \tag{18}$$

- Here, vol_a here denotes the standard a-dimensional Euclidean volume multiplied by a normalization constant, and

$$a = r - \dim T \tag{19}$$

- Here r is the number of positive roots of $SU(3)$. Notice that in this case $a = 1$.

- Therefore $\int_{\mathcal{M}} e^{i\omega}$ can also be expressed as

$$2\pi i \sum_{w_2'\in W} \sum_{w_3'\in W} sgn(w_2')sgn(w_3') \sum_{w_1\in W} sgn(w_1) H_{\underline{\beta}}(w_1\cdot(\xi_1 + w_2'\cdot\xi_2 + w_3'\cdot\xi_3)). \tag{20}$$

- By letting $w_2' = w_1^{-1} w_2,\ w_3' = w_1^{-1} w_3$, we then obtain

$$\int_{\mathcal{M}} e^{i\omega} = 2\pi i \sum_{\underline{w} \in W^3} sgn(\underline{w}) H_{\underline{\beta}}(\mu(\underline{w} \cdot \underline{\xi})). \tag{21}$$

So we obtain the volume formula for triple reduced products for $G = SU(3)$:

Theorem:

$$\mathrm{S}Vol(\mathcal{M}(\underline{\xi})) = \sum_{\underline{w} \in W^3} \mathrm{s}gn(\underline{w}) H_{\underline{\beta}}(\mu(\underline{w} \cdot \underline{\xi})). \tag{22}$$

Here, $H_{\underline{\beta}} : \mathfrak{t}^* \to \mathbf{R}$ is called the Duistermaat-Heckman function. For a general semisimple compact connected Lie group G, it can be defined as follows:

- **Definition:**

$$H_{\underline{\beta}}(\xi) = \mathrm{vol}_a \left\{ (s_1, \cdots, s_r) \ : \ s_i \geq 0, \quad \sum_{i=1}^{r} s_i \beta_i = \xi \right\} \tag{23}$$

- $\underline{\beta} = (\beta_1, \cdots, \beta_r)$ and $\beta_1, \cdots, \beta_r \in \mathfrak{t}^*$ are all the positive roots of G and $a = r - \dim T$.

- For $G = SU(3)$, there are two simple roots β_1, β_2 (with $<\beta_1, \beta_2> = 2\pi/3$) and one additional positive root $\beta_3 = \beta_1 + \beta_2$. Expressing ξ as (ξ_1, ξ_2) where $\xi_j = <\beta_j, \xi>$, we have

$$H_{\underline{\beta}}(\xi) = \xi_1$$

if $\xi_2 > \xi_1$ and

$$H_{\underline{\beta}}(\xi) = \xi_2$$

if $\xi_1 > \xi_2$. Notice that the two definitions agree when $\xi_1 = \xi_2$. The function is continuous along that line, but its first derivatives are not continuous there.

- **Remark:** It is clear from the above definition that $H_{\underline{\beta}}$ is supported in the cone
$$C_{\underline{\beta}} := \left\{ \sum_{i=1}^{r} s_i \beta_i \ : \ s_i \geq 0 \right\} \subseteq \mathfrak{t}^*. \tag{24}$$

- In the case $G = SU(3)$, we have $r = 3$ and
-
$$\beta_1 = H_1,\ \beta_2 = H_2,\ \beta_3 = H_1 + H_2.$$
- If $\xi = (\ell - m)\Omega_1 + m\Omega_2 = (\ell - m) \cdot (2H_1 + H_2)/3 + m \cdot (H_1 + 2H_2)/3$, then we obtain:
$$H_{\underline{\beta}}(\xi) = \kappa \cdot \max\left\{ \min\left\{ \frac{2}{3}\ell - \frac{1}{3}m,\ \frac{1}{3}\ell + \frac{1}{3}m \right\},\ 0 \right\} \tag{25}$$
where κ is a normalization constant.
- We fix the basis $\{\Omega_1, \Omega_2 - \Omega_1\}$ for $\mathfrak{t}$. Then, each $\xi_i = (\ell_i - m_i)\Omega_1 + m_i\Omega_2 \in \mathfrak{t}$ has (ℓ_i, m_i) as its coordinates in this basis. Hence, $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$ can be represented in this basis by the vector
$$(\ell_1, \ell_2, \ell_3, m_1, m_2, m_3) \in \mathbb{R}^6. \tag{26}$$

Hence, the symplectic volume of a triple reduced product for $G = SU(3)$ can be computed explicitly by the following formula:

- **Theorem:**

$$\mathrm{SV}ol(l_1, l_2, l_3, m_1, m_2, m_3) = \tag{27}$$
$$\kappa \sum_{i,j,k=0}^{5} (-1)^{i+j+k} \max\left\{\min\left\{(\frac{2}{3}\pi_1 - \frac{1}{3}\pi_2)(P_{ijk}),\right.\right.$$
$$\left.\left.(\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2)(P_{ijk})\right\},\ 0\right\}$$

- Here,

$$P_{ijk}(l_1, l_2, l_3, m_1, m_2, m_3) = v_i \cdot \begin{pmatrix} l_1 \\ m_1 \end{pmatrix} + v_j \cdot \begin{pmatrix} l_2 \\ m_2 \end{pmatrix} + v_k \cdot \begin{pmatrix} l_3 \\ m_3 \end{pmatrix} \tag{28}$$

and $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the standard projections to the first and second coordinates, respectively. The notations v_i, v_j, v_k denote elements of the Weyl group W.

4. Generalizations

Symplectic volume of triple reduced products for general semisimple compact connected Lie group G

- Our method applies to any semisimple compact connected Lie group G. Therefore the above theorems still hold in this more general situation.
- The set of positive roots is now different and the Duistermaat-Heckman function $H_{\underline{\beta}}$ should be replaced by the general one above.

- **Symplectic volume of N-fold reduced products for general semisimple compact connected Lie group G**
- We can also generalize our results from the triple reduced product (symplectic quotient of product of three orbits) to the N-fold reduced product (symplectic quotient of product of N orbits). The formulas are similar, although we no longer get a piecewise linear function (the formulas are piecewise polynomial).
- **Theorem:**

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{\underline{w} \in W^N} sgn(\underline{w}) \int_{X \in \mathbf{t}} \frac{e^{i\langle \mu(\underline{w} \cdot \underline{\xi}), X \rangle}}{\varpi^{N-2}(X)} \, dX \tag{29}$$

- where $\underline{\xi} = (\xi_1, \cdots, \xi_N), \underline{w} = (w_1, \cdots, w_N) \in W^N$ and

$$\varpi(X) = \prod_{\alpha} \langle \alpha, X \rangle \tag{30}$$

where α runs over all the positive roots of G.

- **Proof:** In this case, the equivariant Euler class is

$$\mathrm{e}_{\underline{w}\cdot\underline{\xi}}(X) = (\mathrm{s}gn(\underline{w}))^N \varpi^N(X). \tag{31}$$

In addition, the symplectic volume of $\mathcal{M}$ can be computed by a similar formula involving Duistermaat-Heckman functions:

- **Theorem:**

$$\mathrm{S}Vol(\mathcal{M}) = \sum_{\underline{w}\in W^N} \mathrm{s}gn(\underline{w}) H_{(N-2)\cdot\underline{\beta}}(\mu(\underline{w}\cdot\underline{\xi})) \tag{32}$$

Here, $\underline{\beta} = (\beta_1, \cdots, \beta_r)$ with $\beta_1, \cdots, \beta_r$ being all the positive roots of G and the Duistermaat-Heckman function $H_{(N-2)\cdot\underline{\beta}}$ is defined as follows:

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$$H_{(N-2)\cdot\underline{\beta}}(\xi) := \mathrm{vol}_a \left\{ (s_1^{(1)}, \cdots, s_r^{(1)}, \cdots, s_1^{(N-2)}, \cdots, s_r^{(N-2)}) \ : \ s_i^{(j)} \geq 0 \text{ for all } i \text{ and } j \text{ and } \sum_{j=1}^{N-2}\sum_{i=1}^{r} s_i^{(j)}\beta_i = \xi \right\} \tag{33}$$

where r is the number of positive roots of G and $a = (N-2)\cdot r - \dim T$.

- **Remark:** Notice that here the Duistermaat-Heckman function is piecewise polynomial.

5. Equivariant cohomology of coadjoint orbits

Let us review the equivariant cohomology of an orbit. For an adjoint orbit homeomorphic to G/T. we see that the cohomology is generated multiplicatively by the first Chern classes of line bundles L_β over the orbit, where

$$L_\beta = G \times_{T,\beta} \mathbf{C}$$

where we write the orbit as G/T and the equivalence relation as

$$(g, z) \sim (gt, \beta(t)^{-1} z)$$

for $g \in G$, $t \in T$, $z \in \mathbf{C}$ and a weight $\beta \in \mathrm{Hom}(T, U(1))$.

Example: For $G = SU(n)$, a proof is in Fulton [F97] Chapter 20.3.

For general Lie groups this result is Theorem 5 and Theorem 11 in Loring Tu's article [TU10]:

Tu, Theorem 5: The ordinary cohomology of G/T is the quotient of the ring of polynomials on $\mathbf{t}$ by the Weyl invariant polynomials of positive degree.

Tu, Theorem 11: The T-equivariant cohomology ring of G/T under the action of T on G/T by left multiplication is

$$H_T^*(G/T) = \frac{\mathbf{Q}[\mathbf{u_1}, \ldots, \mathbf{u}_\ell, \bar{\mathbf{y}}_\mathbf{1}, \ldots, \bar{\mathbf{y}}_\ell]}{\mathcal{J}}$$

where $\mathcal{J}$ is the ideal in $\mathbf{Q}[\mathbf{u_1}, \ldots, \mathbf{u}_\ell, \bar{\mathbf{y}}_\mathbf{1}, \ldots, \bar{\mathbf{y}}_\ell]$ generated by the elements $b(\bar{y}) - b(u)$ for all Weyl invariant polynomials b on $\mathbf{t}$ of positive degree.

6. Intersection Pairings

Write each weight β as

$$\beta(\exp X) = \exp(2\pi B(X))$$

for a linear map $B : \mathbf{t} \to \mathbf{R}$ which sends the integer lattice (the kernel of the exponential map) to $\mathbb{Z}$. Here we have used the exponential map $\exp : \mathbf{t} \to T$. The equivariant first Chern class of the line bundle L_β is denoted

$$c_1^{eq}(L_\beta).$$

Its restriction to an isolated fixed point F of the T action is
$c_1^{eq}(L_\beta)|_F = c_1(L_\beta)_F + B(X)$.

The restriction of this equivariant first Chern class to a component F of the fixed point set is $B(X)$.

By functoriality of characteristic classes, we have

$$(\pi_j)^*(c_1(L_j)) = c_1(\pi_j^* L_j)$$

where

$$\pi_j : \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N} \to \mathcal{O}_{\xi_j}$$

is projection on the j-th orbit, and L_j is a line bundle over $\mathcal{O}_{\xi_j}$.

We then have

Theorem: Let $\mathcal{M}$ be as above, and let ζ be a G-equivariant cohomology class on $\mathcal{M}$. Let $\kappa : H^*_G(\mathcal{M}) \to H^*(\mathcal{M})$ be the Kirwan map. We have

$$\int_{\mathcal{M}} e^{i\omega}\kappa(\zeta) = Res \sum_{\underline{w} \in W^N} \mathrm{sgn}(\underline{w}) \frac{e^{i\langle \mu_T(\underline{w}\cdot\underline{\xi}), X\rangle}\zeta(X)|_{\underline{w}\cdot\underline{\xi}}}{\varpi(X)^{N-2}} dX. \tag{34}$$

Notice that the residue map Res depends on the choice of a cone in $\mathbf{t}$, and only those fixed points whose moment map image lies in this cone give a nonzero contribution.

6.1 Intersection pairings in reduced spaces of products of orbits

The Atiyah-Bott-Berline-Vergne localization formula leads to the following (see [JK95], Theorem 8.1):

$$\int_{M_{\mathrm{red}}} \kappa(\alpha) = \mathrm{Res} \sum_F \alpha_{m/2}(X) \frac{e^{i\mu_X(F)}}{e_F(X)}.$$

$$\int_{M_{\mathrm{red}}} \kappa(\alpha) = \mathrm{Res} \sum_{w \in W} e^{i(w\lambda, X)} \mathrm{sgn}(w) \frac{1}{\left(\varpi(X)\right)^{N-2}}. \tag{35}$$

The above equation is the meaning of the integral over $\mathbf{t}$ in the earlier equation, whose definition is given in [GLS] and developed further in [JK95] The symbol Res (the residue) is defined in [JK95], Theorem 8.1. See also [JK97], Proposition 3.2. The residue has several equivalent definitions (as outlined in [JK97]). One of these definitions characterizes the residue as an iteration of one-variable residues.

In the case when M is the product of N adjoint orbits when

$$\alpha = \exp(i\bar{\omega})$$

is the equivalent extension of the symplectic volume form, and

$$\kappa(\alpha) = e^{i\omega_{\rm red}}$$

is the symplectic volume form on $M_{\rm red}$. The above theorem may be expressed as follows.

In the above notation, we have the following:

Theorem: Let $\mathcal{M}$ be as above, and let ζ be a G-equivariant cohomology class on $\mathcal{M}$. Let $\kappa : H^*_G(\mathcal{M}) \to H^*(\mathcal{M})$ be the Kirwan map. We have

$$\int_{\mathcal{M}} e^{i\omega}\kappa(\zeta) = \sum_{\underline{w}\in W^N} \mathrm{sgn}(\underline{w}) \int_{X\in \mathbf{t}} \frac{e^{i\left\langle \mu_T(\underline{w}\cdot\underline{\xi}),X\right\rangle}\zeta(X)|_{\underline{w}\cdot\underline{\xi}}}{\varpi(X)^{N-2}}\, dX. \tag{36}$$

$$= \mathrm{Res} \sum_{\underline{w}\in W^N} e^{i(\underline{w}\cdot\underline{\xi},X)}\mathrm{sgn}(\underline{w})\frac{\zeta(X)|_{\underline{w}\cdot\underline{\xi}}}{\big(\varpi(X)\big)^{N-2}}. \tag{37}$$

Here $\zeta(X)$ is a product of powers of a collection of equivariant first Chern classes $(c_1^{\rm eq}(L_{\beta_\ell}(X))^{n_\ell}$ where the index ℓ runs from 1 to N if we are considering the reduced space of the product of N orbits and n_ℓ is a nonnegative integer, and the weight of the ℓ-th line bundle is β_ℓ with associated linear map B_ℓ. The restriction of ζ to the fixed point set of the T action is

$$\prod_{\ell} (B_\ell(X))^{n_\ell} .$$

7. Riemann-Roch number of reduced space of coadjoint orbits

By Grothendieck-Riemann-Roch, the Riemann-Roch number of a manifold M is given by

$$\int_M \mathrm{ch}(L^k)\mathrm{Td}(M)$$

where ch is the Chern character and Td is the Todd class.

We are assuming that L is the prequantum line bundle, so this is equal to $\int_M e^{ik\omega}\mathrm{Td}(M)$ where

$$\mathrm{Td}(M) = \prod_\gamma \frac{\gamma}{1 - e^{-\gamma}}$$

in terms of the Chern roots γ. Note that the equivariant Chern roots for the action of T at a fixed point are given by the roots, because the roots specify the action of the maximal torus on the tangent space to the orbit at a fixed point of the T action.

Since the Todd class is multiplicative, it follows from the residue formula that when M is the reduced space of $\prod_j \mathcal{O}(\xi_j)$,

$$RR(M) = Res \prod_{j=1}^{N} \sum_{w_j \in W} \sum_F \prod_{i=1}^{N} e^{i < \sum_{j=1}^{N} > w_j \xi_j, X >} > \prod_{j=1}^{N} \prod_{\gamma_j} \frac{\gamma_j(X)}{1 - e^{-\gamma_j(X)}}.$$

We are using the fact that the restriction of the equivariant Chern roots of a coadjoint orbit to a fixed point F of the action of T are the roots $\gamma(X)$ evaluated on the parameter $X \in \mathbf{t}$.