Talk

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Laughlin states on Riemann surfaces

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Laughlin state

$$\Psi_L(z_1, \dots, z_N) = C \cdot \prod_{n<m}^{N} (z_n - z_m)^\beta \cdot e^{-\frac{B}{4} \sum_{n=1}^{N} |z_n|^2}$$

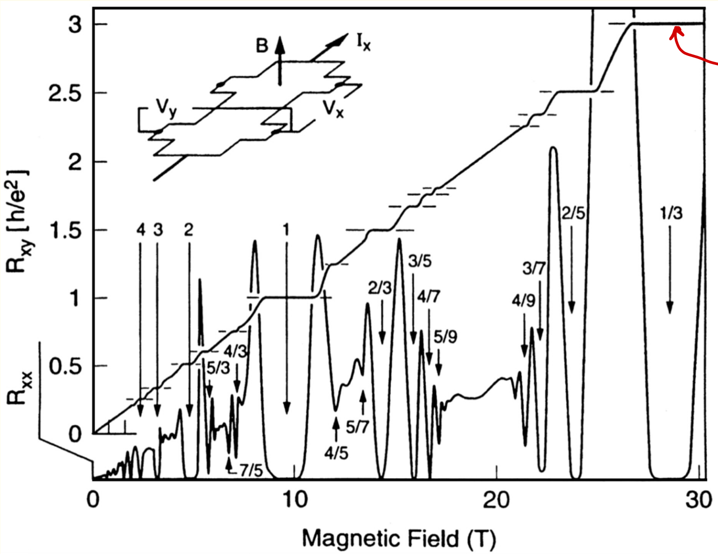
$\{z_n\} \in \mathbb{C}^N$

$\beta \in \mathbb{Z}_+$ $\quad$ $1/\beta$ is "filling fraction"

$B > 0$ $\quad$ "magnetic field"

Quantum Hall effect (QHE)

Precise quantization of Hall conductance $\sigma_H = \frac{1}{R_{xy}}$



Laughlin state corresponds to this plateau

$\sigma_H = 1/\beta$

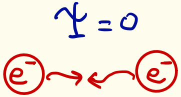
Fractional QHE

Strongly-interacting (via Coulomb forces) system.

Laughlin 1983: Assign a trial wave function ("state") to each plateau.

$$\Psi_L(z_1,\dots,z_N) = C \cdot \prod_{n<m}^{N} (z_n - z_m)^{\beta} \cdot e^{-\frac{B}{4}\sum_{n=1}^{N}|z_n|^2}$$

* holomorphic



* vanishing conditions

* $\beta = 1$: Slater determinant (free particles)

$$\prod_{n<m}^{N} (z_n - z_m) = \det z_m^{\,n-1}$$

$\beta = 1$ case is also called "Integer QH State"

Another QHE state

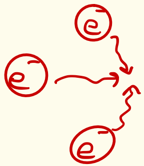
Moore-Read 1991:

$$\Psi_{MR}(z_1, \ldots, z_N) = c \cdot \mathrm{Pf}\left(\frac{1}{z_n - z_m}\right) \cdot \prod_{n<m}^{N} (z_n - z_m) \cdot e^{-\frac{B}{4} \sum_n |z_n|^2}$$

Pfaffian of anti-sym. matrix

$$\mathrm{Pf}(M) = \sqrt{\det M}$$

$$\sigma_H = 5/2$$



$\Psi = 0$

Goal

Define QH states (e.g. Laughlin state) on compact Riemann surfaces and study how do they depend on geometric data (genus, metric, moduli...) for large N.

Haldane 1983	$g=0$
Haldane-Rezayi 1985	$g=1$
Wen-Niu 1991	$g>1$
Avron-Seiler-Zograf 1995,	$\beta=1$, $\mathrm{Jac}(\Sigma)$, $\mathcal{M}_{1,1}$
N. Read 2009	$\beta>1$, $\mathcal{M}_{1,1}$

Normalization and large N

QM wave functions shall be normalized

$$Z = \int_{\mathbb{C}^N} |\Psi_L(z_1,\dots,z_N)|^2 \prod_{n=1}^{N} d^2 z_n$$

$$= \frac{1}{N!} \int_{\mathbb{C}^N} \exp\left[-\frac{\beta}{2} \sum_n |z_n|^2 + \beta \sum_{n \neq m} \log |z_n - z_m|\right] \prod_{n=1}^{N} d^2 z_n$$

2D Coulomb gas partition function.

More generally,

$$Z = \int_{\mathbb{C}^N} \exp\left\{ -N \sum_{n=1}^{N} V(z_n, \bar{z}_n) + \beta \sum_{n \neq m} \log |z_n - z_m| \right\} \cdot \prod_{n=1}^{N} d^2 z_n$$

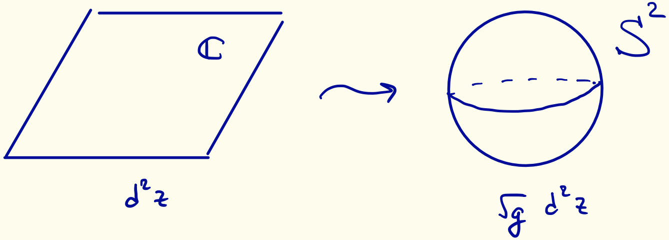
geometric spin ($s=1$ in pure Coulomb gas)

$$V = \phi(z, \bar{z}) - \frac{1-s}{N} \log \sqrt{g}(z, \bar{z})$$

volume form $\sqrt{g}\, d^2 z$ on $\mathbb{C}$

"magnetic potential"

$B = N \Delta \phi > 0$



Coulomb gas

Random normal/complex matrices

Beta ensembles

Math result

Thm Leblé-Serfaty 2015
Bauerschmidt et al 2016

$$\log Z = -\beta N^2 I_V(\mu_V) + \frac{\beta}{2} N \log N - N C(\beta) - N\left(1-\frac{\beta}{2}\right)\int_{\mathbb{C}} \mu_V \log\frac{\mu_V}{\mu_0} + o(N)$$

where $I_V = -\iint_{\mathbb{C}\times\mathbb{C}} \log|z-w|\, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} V\, d\mu$

and μ_V its unique minimizer ("equilibrium measure")

Physics result

Can, Laskin, Wiegmann 2014
F. Ferrari, SK 2014

Loop equations
Free field

$$\log Z = -\beta N^2 I_V(\mu_V) - N\left(s - \frac{\beta}{2}\right)\int_\Sigma \mu_V \log\frac{\mu_V}{\mu_0}$$

$$- \frac{c_H}{12}\int_\Sigma \left(\left|\partial \log\frac{\mu_V}{\mu_0}\right|^2 - 2\log\frac{\mu_V}{\mu_0}\,\partial\bar\partial\log\mu_0\right) + \text{const} + R_{1/N}$$

Liouville functional

remainder terms

$$c_H = 1 - 3\left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}}\right)^2 \qquad (s = 1 \text{ in pure Coulomb gas})$$

Coefficients in this expansion are of interest.

$R_{1/N}$ is a local funct. of B and curvature R of g.

Free field representation

CFT w/ background charge & magnetic field

$$S(\sigma) = \frac{1}{2\pi} \int_\Sigma \left(|\nabla \sigma|_g^2 + \frac{i}{4} Q \sigma R + \frac{i}{\sqrt{\beta}} \sigma B \right) dv_g$$

$$Q = \sqrt{\beta} - \frac{2S}{\sqrt{\beta}}$$

$$E[\ldots] = \int_{\mathcal{E}} \ldots e^{-S(\sigma)} d_g \sigma$$

$$|\Psi_L(z_1, \ldots, z_N)|^2 = E\left[e^{i\sqrt{\beta}\sigma(z_1)} \cdots e^{i\sqrt{\beta}\sigma(z_N)} \right]$$

Moore-Read 1991

$$E\left[e^{i\alpha_1 \sigma(z)} e^{i\alpha_2 \sigma(w)} \right] \simeq e^{-\alpha_1\alpha_2 G(z,w)} \simeq |z-w|^{\alpha_1\alpha_2}$$

Remainder term

$$R_{1/N} = \log \int d_g\sigma \; e^{-S(\sigma) + N \log \int_\Sigma e^{i\sqrt{\beta}\sigma} dv_g}$$

$$- \log \int d_{g_0}\sigma \; e^{-S(\sigma) + N \log \int_\Sigma e^{i\sqrt{\beta}\sigma} dv_{g_0}}$$

$$= \log E_g \left(\int e^{i\sqrt{\beta}\sigma} dv_g \right)^N - \log E_{g_0} \left(\int e^{i\sqrt{\beta}\sigma} dv_{g_0} \right)^N$$

Conjecture Ferrari-SK

$$R_{1/N} = O(1/N)$$

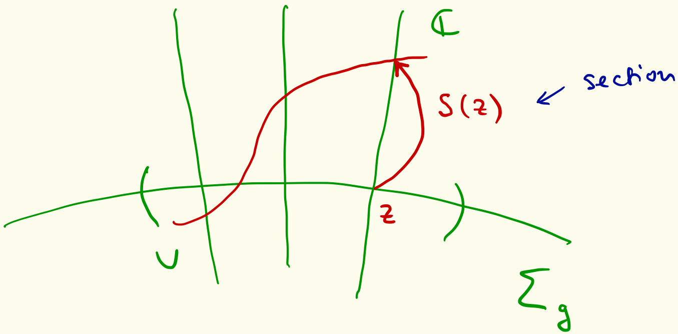
as $N \to \infty$

QH states on Riemann surfaces

Recall, that the one-particle wave functions on $\mathbb{C}$ are $\psi_n = z^n e^{-\frac{B}{4}|z|^2}$.

Polynomials on $\mathbb{C}$ $\quad z^n \rightsquigarrow S_n(z)$, holomorphic sections of a <u>holomorphic line bundle</u> L of degree $N_\varphi \in \mathbb{Z}_+$.

(L, h), $R_h = -\partial\bar\partial \log h > 0 \quad \sim$ magnetic field



+ holomorphic transition functions t_{UV} on $U \cap V$

$L \sim \mathcal{O}(D)$

Integer QH state (Slater determinant)

Consider $\Sigma^N = \underbrace{\Sigma \times \dots \times \Sigma}_{N}$, $N = \dim H^0(\Sigma, L)$

$$\Psi(z_1, \dots, z_N) = \det S_n(z_m) \Big|_{n,m=1}^{N}$$

Its L^2-norm is given by

$$Z = \int_{\Sigma^N} \| \det S_n(z_m) \|_h^2 \cdot \prod_{n=1}^{N} dv_g(z_n)$$

Prop Let $B = N_\phi$, g is an arbitrary smooth metric on Σ, parameterized as $g = g_0 + \partial\bar\partial\phi$ and hermitian metric $h = h_0 e^{-\phi}$. Then the following asymptotic expansion holds

SK 2014

SK-Ma-Marinescu -Wiegmann 2017

$$\log Z = -N_\phi^2 S_{AY}(g_0, \phi) + \frac{1}{2} N_\phi S_M(g_0,\phi) + \frac{1}{6} S_L(g_0,\phi) + O(1/N_\phi)$$

where

$$S_{AY} = \frac{1}{2\pi}\int_\Sigma \left(|\partial\phi|^2_{g_0} + \phi \right) dv_{g_0} \qquad \text{Aubin-Yau}$$

$$S_M = \frac{1}{2\pi}\int_\Sigma \left(-\frac{1}{2}\phi R_0 + \frac{\sqrt{g}}{\sqrt{g_0}} \log\frac{\sqrt{g}}{\sqrt{g_0}} \right) dv_{g_0} \qquad \text{Mabuchi}$$

$$S_L = \frac{1}{2\pi}\int_\Sigma \left(\left|\partial \log \frac{\sqrt{g}}{\sqrt{g_0}}\right|^2_{g_0} + R_0 \log\frac{\sqrt{g}}{\sqrt{g_0}} \right) dv_{g_0} \qquad \text{Liouville}$$

Proof Variational f-la

$$\delta \log Z = -\frac{1}{2\pi} \int_\Sigma \left(N_\phi B_{N_\phi} \cdot \delta\phi - \frac{1}{2} \Delta B_{N_\phi} \cdot \delta\phi \right) \sqrt{g}\, d^2 z$$

where Bergman kernel for the $H^0(\Sigma, L)$

$$B_{N_\phi}(z, \bar{z}) = \sum_{n=1}^{N} \| s_n(z) \|_h^2 \simeq N_\phi + \frac{1}{2} R(g) + O(1/N_\phi)$$

Complete asymptotic expansion at large N_ϕ

(Boutet de Monvel - Sjostrand, Zelditch, Catlin, ...)

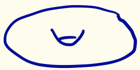
$\square$

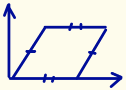
Laughlin states on $\Sigma_{g>1}$

* Haldane-Rezayi (85)

  β- degeneracy of Laughlin states on torus

  Breaking of translation symmetry?

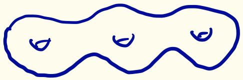




* Wen-Niu (89)

  Topological degeneracy

  β^g Laughlin states on genus-g Σ (conjecture).



"Topological phases of matter"

Definition of Laughlin states for $g>1$

One may choose

$$|\Psi_L(z_1,\dots,z_N)|^2 = e^{-\beta \sum_{n<m} G(z_n, z_m)}$$

Locally looks like $\prod_{n<m} |z_n - z_m|^{2\beta}$,

but no determinant representation at $\beta = 1$

$$e^{-\sum_{n<m} G(z_n, z_m)} \neq \det S_n(z_m)$$

$N-1$ zeroes in z_1

$N-1+g$ zeroes in z_1

(Riemann - Roch thm)

Laughlin states on $\Sigma_{g>1}$

$$\Psi(z_1,\dots,z_N) = \prod_{n<m} (z_n - z_m)^\beta \, e^{-\frac{B}{4}\sum_n |z_n|^2}$$

$$\beta \in \mathbb{Z}_+$$

Def Consider holomorphic line bundle (L,h) of degree N_φ.

Consider $\Sigma^N = \Sigma \times \dots \times \Sigma$

$N = \frac{1}{\beta} N_\varphi + 1 - g$ (assume $\beta \mid N_\varphi$)

* $\pi_n \Psi \in H^0(\Sigma, L)$ (restriction to n-th factor in $\Sigma \times \dots \times \Sigma$)

* $\pi_{nm} \Psi \simeq (z_n - z_m)^\beta$, near diagonal $(z_n \sim z_m)$

* Ψ is completely symm. (antisymm.) for $\beta \in$ even (odd)

Thm For $N \geq g$, the following is the basis of the vector space of Laughlin states

(Wen-Niu 1990 conjecture)

$r = (1, \dots \beta)^g$

$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_{n=1}^{N} z_n - \beta\Delta - \beta D, \beta\tau\right) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} \sigma(z_n)^{\frac{1}{g}N_\phi - \beta}}$$

SK, Commun. Math. Phys. 2019; SK-Zvonkine to appear

Proof

1. Prime-form on $\Sigma \times \Sigma$

$$E(z, y) = \frac{\theta\left(\int_z^y \omega - \delta, \tau\right)}{\sqrt{\omega_\delta(z)}\ \sqrt{\omega_\delta(y)}} \simeq \frac{z-y}{\sqrt{dz}\ \sqrt{dy}}$$

$\exists\ \delta \in (\mathbb{Z}/_2\mathbb{Z})^g$ (non-singular θ-characteristic), s.t.

$\operatorname{div} \omega_\delta = 2\, D_{g-1}$

2. $\beta = 1$ $N = N_\varphi + 1 - g$ $\{s_n\} \in H^0(\Sigma, L)$

$$\Psi = \det s_n(z_m)$$

$$= \prod_{n<m}^{N} E(z_n, z_m) \cdot f(z_1, \ldots, z_N)$$

Fay identity (1973)

$$f(z_1, \ldots, z_N) = C \cdot \Theta\left(\sum_n z_n - \Delta - D, \tau\right)$$

so f is a symmetric section of $f_L^* \hat{\Theta} \to \mathrm{Sym}^N \Sigma$

$$f_L : \quad z_1, \ldots, z_N \mapsto \sum_n z_n - \Delta - D \in \mathrm{Jac}(\Sigma)$$

(Proof by comparison of zeros on both sides)

3. $\Psi \sim \prod_{n<m}^{N} \left(E(z_n, z_m)\right)^{\beta} \cdot f_\beta(z_1, \dots, z_N)$

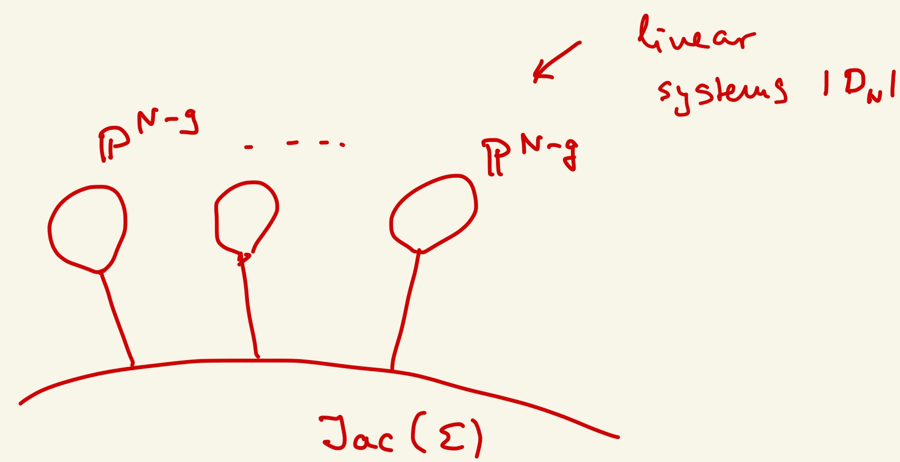
Let $L = \beta L'$ and $\{s_n\} \in H^0(\Sigma, L')$

Then $\left(\det s_n(z_m)\right)^{\otimes \beta}$ is a Laughlin state

$\Rightarrow$ $f_\beta(z_1, \dots, z_N)$ is a symmetric section of

$$\left(f_L^* \hat{\Theta}\right)^{\otimes \beta} = f_L^* \hat{\Theta}^{\beta}$$

$f_L : \mathrm{Sym}^N \Sigma \longrightarrow$



There are β^g independent sections of $\hat{\Theta}^\beta$, given by order-β Θ-functions

$$\Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}(\beta Y, \beta\tau) \qquad Y \in \mathbb{C}^g/\Lambda$$

$$r = (1, \dots, \beta)^g$$

$$\Psi_r \sim \prod_{n<m}^{N} \left(E(z_n, z_m)\right)^\beta \cdot \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum z_n - \beta\Delta - \beta D, \beta\tau\right)$$

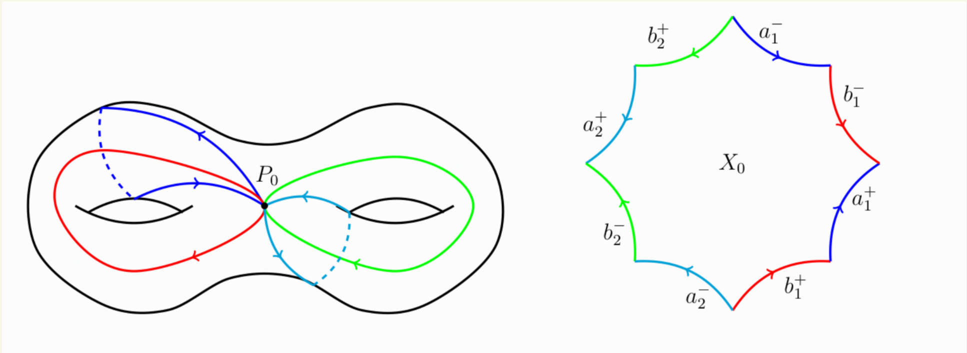
for $N \geq g$ (by Jacobi thm. $I(D_g) = \mathrm{Jac}(\Sigma)$)

4. So far Ψ_r is a degree $-\frac{1}{2}\beta(N-1)$ form on each Σ.

$$G(p) = \exp\Big(-\sum_{j=1}^{g} \int_{a_k} \omega_k(z) \log E(z,p)\Big)$$

$\frac{g}{2}$ - form with no poles / zeroes

$G(p)$ lives on X_0



$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\Big(\beta \sum_{n=1}^{N} z_n - \beta\Delta - \beta D, \beta\tau\Big) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} G(z_n)^{\frac{1}{g}\beta(N-1)}}$$

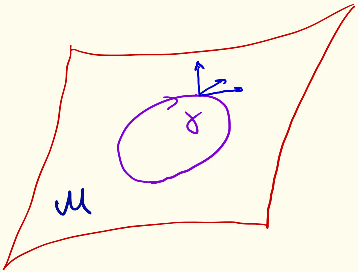
Geometric adiabatic transport

QHE wave functions are typically degenerate

(β^g Laughlin states on genus-g surface) and depend on

parameter spaces $\mathcal{M}$ (e.g. moduli space $\mathcal{M}_{g,n}$)

Thus we have a <u>Hilbert bundle</u> $V_{QH} \to \mathcal{M}$

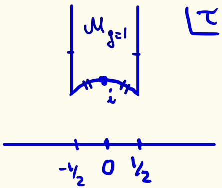


adiabatic transport:

$$\Psi_r \to U_{rr'}(\gamma)\,\Psi_{r'}$$

$\uparrow$

holonomy matrix



Conjecture

N.Read 2008 (for $g>0$) "holonomy equals monodromy"

V_{QH} is projectively flat (at least as $N \to \infty$)

(i.e. Berry curvature is $\mathcal{R} = c \cdot \mathbb{1}$, or equivalently

adiabatic transport is independent of the path in $\mathcal{M}$,

up to $U(1)$ phase)

Berry connection $\nabla : \Gamma(V) \to \Omega^1(V)$

$$\partial_y \langle \psi , \psi' \rangle_{L^2} = \langle \nabla \psi, \psi' \rangle_{L^2} + \langle \psi, \nabla \psi' \rangle_{L^2} \qquad y \in \mathcal{M} .$$

Projective flatness in CFT :

Axelrod-della Pietra-Witten'90
Hitchin'90

* Laughlin and Pfaffian states are projectively flat on $\mathcal{M}_{1,1}$

$$\nabla^H \Psi_L = 0 \qquad \nabla^H = 4\pi i N_\phi \partial_\tau - \sum_{n=1}^{N} \partial^2_{z_n} + 2\beta(\beta-1) \sum_{n<m} \wp(z_n - z_m)$$

w/ N. Nemkov

* Is $H^0(\Sigma, L^d)$ projectively flat over $\mathcal{M}_{g,n}$, $\mathrm{Pic}_d(\Sigma)$?

The End