Trace formula for the magnetic Laplacian

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Introduction

Trace formulas are quantitative relations between the eigenvalues of a quantum Hamiltonian and dynamical invariants of the corresponding classical dynamical system:

- In quantum mechanics, the semiclassical Gutzwiller formula for a Schrödinger operator $H_h = -\hbar^2\Delta + V(x)$ in $\mathbb{R}^n$.
- The Duistermaat-Guillemin trace formula for the Laplacian Δ_g of a compact Riemannian manifold (M, g)
(homogeneous quantization, closed geodesics).
- The Selberg trace formula.

The goal:

I am going to discuss trace formulas for magnetic fields related with closed magnetic geodesics, more precisely, the trace formula for the magnetic Laplacian on a compact Riemannian manifold
(the Guillemin-Uribe trace formula for reduced Hamiltonian systems).

The data

(M, g) a compact Riemannian manifold of dimension n.
F an arbitrary closed 2-form on M (a magnetic field).

Define the twisted symplectic form on the phase space $X = T^*M$:

$$\Omega = \Omega_0 + \pi_M^* F,$$

where Ω_0 is the canonical symplectic form on T^*M and $\pi_M : T^*M \to M$ is the bundle map.
In local coordinates $(x^1, x^2, \dots, x^n, p_1, p_2, \dots, p_n)$ on T^*M,

$$\Omega = \sum_{j=1}^{n} dp_j \wedge dx^j + \sum_{j,k=1}^{n} F_{jk} dx^j \wedge dx^k$$

where $F = \sum_{j,k=1}^{n} F_{jk} dx^j \wedge dx^k$.

The magnetic geodesic flow $\phi^t : T^*M \to T^*M$ associated with (g, F) is the Hamiltonian flow with respect to the twisted symplectic form Ω of the Hamiltonian

$$\mathcal{H}(x,p) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk} p_j p_k.$$

The Hamilton equations of the magnetic flow ϕ^t:

$$\begin{aligned}\frac{dx^j}{dt} &= \sum_{k=1}^{n} g^{jk} p_k,\\ \frac{dp_j}{dt} &= -\frac{1}{2}\sum_{k,\ell=1}^{n} \frac{\partial g^{k\ell}}{\partial x^j} p_k p_\ell + \sum_{k,\ell=1}^{n} F_{jk} g^{k\ell} p_\ell, \quad j = 1, \ldots, n.\end{aligned}$$

The motion of a charged particle on M in the external magnetic field F.

A quantum line bundle is a triple (L, h^L, ∇^L):

- L a complex line bundle on M;
- h^L a Hermitian structure in the fibers of L;
- $\nabla^L : C^\infty(M, L) \to C^\infty(M, T^*M \otimes L)$ a Hermitian connection (a magnetic potential),

satisfying the condition:

$$F = iR^L.$$

where R^L is the curvature of ∇^L:

$$R^L(U, V) = \nabla^L_U \nabla^L_V - \nabla^L_V \nabla^L_U - \nabla^L_{[U,V]}, \quad U, V \in TM.$$

Quantization condition:

A quantum line bundle (L, h^L, ∇^L) exists $\Leftrightarrow [F] \in H^2(M, 2\pi\mathbb{Z})$.

The magnetic Laplacian Δ^L is the Bochner Laplacian associated with the quantum line bundle (L, h^L, ∇^L):

$$\Delta^L = (\nabla^L)^*\nabla^L : C^\infty(M, L) \to C^\infty(M, L).$$

Given a Riemannian metric g on M, we have L^2-inner products on $C^\infty(M, L)$ and $C^\infty(M, T^*M \otimes L)$:

$$(s, s')_{L^2(M,L)} = \int_M (s(z), s'(z))_{h^L} dv_g(z), \quad s, s' \in C^\infty(M, L).$$

The formally adjoint operator of $\nabla^L : C^\infty(M, L) \to C^\infty(M, T^*M \otimes L)$

$$(\nabla^L)^* : C^\infty(M, T^*M \otimes L) \to C^\infty(M, L).$$

For $s \in C^\infty(M, L), s' \in C^\infty(M, T^*M \otimes L)$:

$$(\nabla^L s, s')_{L^2(M,T^*M\otimes L)} = (s, (\nabla^L)^* s')_{L^2(M,L)}.$$

Example

- $M = \mathbb{R}^n$, $L = M \times \mathbb{C}$ the trivial Hermitian line bundle.
- $\nabla^L = d + \Gamma$, where $\Gamma = -i\mathbf{A}$, $\mathbf{A} = \sum_{j=1}^n A_j(x)dx_j$ is a real-valued one form (the magnetic potential).
- The curvature $R^L = d\Gamma = -id\mathbf{A}$.
- The magnetic field $F = \mathbf{B} := d\mathbf{A}$,

$$\mathbf{B} = \sum_{j,k=1}^n B_{jk}(x)dx_j \wedge dx_k, \quad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}.$$

- The Riemannian metric g is the standard metric on $\mathbb{R}^d$.
- The magnetic Laplacian:

$$\Delta^L = -\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} - iA_j(x)\right)^2.$$

Semiclassical parameter:

- $L^N = L^{\otimes N}$ the N-th tensor power of L.
- Δ^{L^N} the magnetic Laplacian acting on $C^\infty(M, L^N)$:

$$\Delta^{L^N} = -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j} - iNA_j(x)\right)^2 .$$

- Semiclassical limit $N \to \infty \Leftrightarrow \hbar = N^{-1} \to 0$.

$\{\nu_{N,j}, j \in \mathbb{N}\}$ the eigenvalues of Δ^{L^N}:

$$\Delta^{L^N} u_{N,j} = \nu_{N,j} u_{N,j}.$$

Put

$$\lambda_{N,j} = \sqrt{\nu_{N,j} + N^2}.$$

(the eigenvalues of $\sqrt{\Delta^{L^N} + N^2}$).

Fix $E > 1$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Define

$$Y_N(\varphi) = \sum_{j=0}^{\infty} \varphi(\lambda_{N,j} - EN).$$

This is a smoothed version of $\#\{j \in \mathbb{N} : |\lambda_{N,j} - EN| < c\}$, $c > 0$.
Recall that $\lambda_{N,j} = \sqrt{\nu_{N,j} + N^2}$, $\{\nu_{N,j}, j \in \mathbb{N}\}$ the eigenvalues of Δ^{L^N}.

The trace formula (Guillemin-Uribe, 1989)

An asymptotic expansion of $Y_N(\varphi)$ as $N \to \infty$ is expressed in terms of the magnetic geodesic flow at the fixed energy level E under some assumption on the flow.

Following Guillemin-Uribe, we will consider the Hamiltonian

$$H(x,p) = (2\mathcal{H}(x,p)+1)^{1/2} = \left(\sum_{j,k=1}^{n} g^{jk} p_j p_k + 1\right)^{1/2}.$$

Observe that $H(x,p) = E \Leftrightarrow \mathcal{H}(x,p) = \mathcal{E}$, where $\mathcal{E} = \frac{E^2-1}{2}$.

Assumption:

Fix $E > 1$. Then the Hamiltonian flow ϕ of H is clean on the energy level $X_E := H^{-1}(E) \subset T^*M$, a smooth submanifold of T^*M..

If the set of periods of the flow is discrete, then ϕ is clean on X_E if:

- for every period T, the set $\mathcal{P}_T = \{x \in X_E : \phi^T(x) = x\}$ is a smooth manifold,
- at each $x \in \mathcal{P}_T$ its tangent space $T_x\mathcal{P}_T$ is identical with the set of fixed vectors of $d(\phi^T)_x$.

Guillemin-Uribe trace formula

Theorem

For $E > 1$ and $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} Y_N(\varphi) &= \sum_{j=0}^{\infty} \varphi(\lambda_{N,j} - EN) \\ &\sim c_0(N,\varphi)N^d + c_1(N,\varphi)N^{d-1} + c_2(N,\varphi)N^{d-2} + \dots, \quad N \to \infty, \end{aligned}$$

where the coefficients $c_j(N,\varphi)$ are bounded in N.

Recall that $\lambda_{N,j} = \sqrt{\nu_{N,j} + N^2}$, $\{\nu_{N,j}, j \in \mathbb{N}\}$ are the eigenvalues of Δ^{L^N}.

Guillemin-Uribe trace formula

Moreover, we can say the following about the leading coefficient in the expansion, c_0, and the degree $d = d(\varphi)$:

$$Y_N(\varphi) = \sum_{j=0}^{\infty} \varphi(\lambda_{N,j} - EN) \sim \sum_{j=0}^{\infty} c_j(N,\varphi) N^{d-j}, \quad N \to \infty.$$

Theorem (continued)

If 0 is the only period in $supp(\hat{\varphi})$, then $d = n - 1$ and

$$c_0(N,\varphi) = (2\pi)^{-n} \hat{\varphi}(0) \mathrm{Vol}(X_E).$$

Theorem (continued)

Assume:

- there is a unique period T of the flow in the support of $\hat{\varphi}$.
- Let $Y_1, \ldots, Y_r$ be the connected components of $\mathcal{P}_T = \{x \in X_E : \phi^T(x) = x\}$ of maximal dimension k.

Then

$$d = (k-1)/2,$$

and for each j there is a density ν_j on Y_j defined in terms of the classical dynamics:

$$c_0(N, \varphi) = \hat{\varphi}(T) \sum_{r=1}^{r} e^{\pi i m_j/4} e^{-iNS_j} \int_{Y_j} \nu_j,$$

where $m_j = m_{Y_j}$ is the (common) Maslov index of the trajectories in Y_j and S_j is their (common) action.

The action of periodic trajectory

- γ is a periodic trajectory on $X_E \subset T^*M$ of length L;
- $\pi_M \circ \gamma$ the projection of γ to M;
- In the case when $F = dA$ for some real-valued 1-form A, the action S_γ of γ:
$$S_\gamma = \pm L\sqrt{E^2 - 1} + \int_\gamma \pi_M^* A.$$
- In the case of an arbitrary F, the action S_γ of γ is defined modulo multiples of 2π:
$$S_\gamma = \pm L\sqrt{E^2 - 1} + h_A(\gamma).$$

$h_A(\gamma) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the holonomy of $\pi_M \circ \gamma$ with respect to ∇^L.

Constant magnetic field on the two-torus

- $M=\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$.
- $g=dx^2+dy^2$ the Riemannian metric.
- $F=Bdx\wedge dy$ the magnetic field.
- The quantization condition

$$B=2\pi n,\quad n\in\mathbb{Z}.$$

- Sections of a quantum line bundle L over $\mathbb{T}^2$ are identified with functions $u\in C^\infty(\mathbb{R}^2)$ such that

$$u(x+1,y)=e^{iBy}u(x,y),\quad u(x,y+1)=u(x,y).$$

- Put $B = 2\pi$.
- The Hermitian connection on L has the form

$$\nabla^L = d - iA,$$

where A is the magnetic potential: $A = 2\pi x dy$.

- The magnetic form $F = dA = 2\pi dx \wedge dy$.
- The operator

$$\Delta^{L^N} = -\frac{\partial^2}{\partial x^2} - \left(\frac{\partial}{\partial y} - 2\pi N i x\right)^2 .$$

- Its eigenvalues (Landau levels)

$$\nu_{N,j} = 2\pi N(2j+1), \quad j = 0, 1, 2, \ldots .$$

with multiplicity

$$m_{N,j} = N.$$

For $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$Y_N(\varphi) = \sum_{j=0}^{\infty} N\varphi\left(\sqrt{N^2 + 2\pi N(2j+1)} - EN\right).$$

Let us write

$$j = \frac{E^2-1}{4\pi}N - \left\{\frac{E^2-1}{4\pi}N\right\} + n, \quad n \in \mathbb{Z}.$$

Then we get

$$Y_N(\varphi)$$
$$= \sum_{n=-\left[\frac{E^2-1}{4\pi}N\right]}^{\infty} N\varphi\left(\sqrt{E^2N^2 + 2\pi N(2n+1) - 4\pi N\left\{\frac{E^2-1}{4\pi}N\right\}} - EN\right).$$

Recall the Taylor expansion for $f(x)=\sqrt{1+x}-1$ at $x=0$:

$$f(x)=\sqrt{1+x}-1=\frac{1}{2}x-\frac{1}{8}x^2+\sum_{k=3}^{\infty}c_kx^k,$$

where

$$c_k=(-1)^{k-1}\frac{(2k-3)!}{2^{2k-2}k!(k-2)!},\quad k\geqslant 2.$$

For each n, we have the following asymptotic expansion as $p\to+\infty$:

$$\begin{aligned}\lambda_{N,j}-EN=&\frac{1}{E}\left(\pi(2n+1)-2\pi\left\{\frac{E^2-1}{4\pi}N\right\}\right)\\&-\frac{1}{8E^3N}\left(2\pi(2n+1)-4\pi\left\{\frac{E^2-1}{4\pi}N\right\}\right)^2\\&+\sum_{k=3}^{\infty}\frac{c_k}{E^{2k-1}N^{k-1}}\left(2\pi(2n+1)-4\pi\left\{\frac{E^2-1}{4\pi}N\right\}\right)^k.\end{aligned}$$

Using Taylor expansion of φ:

$$\varphi(a_0 + a_1 p^{-1} + a_2 p^{-2} + \dots) \\ = \varphi(a_0) + \varphi'(a_0)a_1 + (\varphi'(a_0)a_2 + \frac{1}{2}\varphi''(a_0)a_1^2)p^{-2} + \dots.$$

we obtain an asymptotic expansion for $Y_N(\varphi)$:

$$Y_N(\varphi) = \sum_{j=0}^{\infty} N\varphi\left(\lambda_{N,j} - EN\right) \\ \sim c_0(N,\varphi)N + c_1(N,\varphi) + c_2(N,\varphi)N^{-1} + \dots, \quad N \to \infty.$$

For the leading term of order N, we get

$$c_0(N,\varphi) = \sum_{n\in\mathbb{Z}} \varphi\left(\frac{\pi(2n+1)}{E} - \frac{2\pi}{E}\left\{\frac{E^2-1}{4\pi}N\right\}\right).$$

By Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(nP+t)=\sum_{k\in\mathbb{Z}}\frac{1}{P}\hat{f}\left(\frac{2\pi k}{P}\right)e^{2\pi i\frac{k}{P}t}.$$

with

$$\hat{f}(k)=\int f(x)e^{-ikx}dx,$$

we get

$$\begin{aligned} c_0(N,\varphi)=&\sum_{k\in\mathbb{Z}}\frac{E}{2\pi}\hat{\varphi}(kE)e^{ik\pi}e^{-2\pi ik\left\{\frac{E^2-1}{4\pi}N\right\}}\\ =&\frac{E}{2\pi}\sum_{k\in\mathbb{Z}}\hat{\varphi}(kE)e^{ik\pi}e^{-ik\frac{E^2-1}{2}N}. \end{aligned}$$

For the next term, we get

$$c_1(N,\varphi)=-\sum_{k\in\mathbb{Z}}\left(\frac{i}{2\pi E^2}\hat{\varphi}'(kE)+\frac{ik}{4\pi E}\hat{\varphi}''(kE)\right)e^{-ik\frac{E^2-1}{2}N}.$$

Contribution of zero

$$c_0(N,\varphi)=\frac{E}{2\pi}\sum_{k\in\mathbb{Z}}\hat\varphi(kE)e^{ik\pi}e^{-ik\frac{E^2-1}{2}N}.$$

If 0 is the only period in *supp*$(\hat\varphi)$ ($k=0$), we have

$$c_0(N,\varphi)=\frac{E}{2\pi}\hat\varphi(0).$$

From the general theorem, we know $d=n-1$ and

$$c_0(N,\varphi)=(2\pi)^{-n}\hat\varphi(0)\mathrm{Vol}(X_E).$$

where

$$X_E=H^{-1}(E)=\{(x,y,p_x,p_y):p_x^2+p_y^2=E^2-1\}\subset T^*M.$$

We have $n=2$, $d=n-1=1$ and

$$\mathrm{Vol}(X_E)=2\pi E.$$

Contributions of periodic trajectories

The magnetic field F on $\mathbb{T}^2$ given by

$$F = Bdx \wedge dy, \quad B \in \mathbb{R}.$$

The Hamiltonian H has the form

$$H(x, y, p_x, p_y) = \left(p_x^2 + p_y^2 + 1\right)^{1/2},$$

and the reduced Hamiltonian system on X_E is given by

$$\dot{x} = \frac{1}{E} p_x, \quad \cdot y = \frac{1}{E} p_y, \quad \dot{p}_x = \frac{1}{E} B p_y, \quad \dot{p}_y = -\frac{1}{E} B p_x.$$

Contributions of periodic trajectories

This system can be easily solved (“Larmor orbits”):

$$
\begin{aligned}
x(t) =& x^0 + \frac{1}{B}p_y^0\left(1-\cos\frac{B}{E}t\right) + \frac{1}{B}p_x^0\sin\frac{B}{E}t,\\
y(t) =& y^0 + \frac{1}{B}p_y^0\sin\frac{B}{E}t - \frac{1}{B}p_x^0\left(1-\cos\frac{B}{E}t\right),\\
p_x(t) =& p_y^0\sin\frac{B}{E}t + p_x^0\cos\frac{B}{E}t,\\
p_y(t) =& p_y^0\cos\frac{B}{E}t - p_x^0\sin\frac{B}{E}t.
\end{aligned}
$$

So each trajectory is periodic with period $T=\frac{2\pi E}{B}$. Its projection γ is the image of a circle of radius $\frac{1}{B}\sqrt{(p_x^0)^2+(p_y^0)^2}$ under the natural projection $f:\mathbb{R}^2\to\mathbb{T}^2$.

Contributions of periodic trajectories

$$c_0(N,\varphi)=\frac{E}{2\pi}\sum_{k\in\mathbb{Z}}\hat{\varphi}(kE)e^{ik\pi}e^{-ik\frac{E^2-1}{2}N}.$$

Assume that there is a unique period T of the flow in the support of $\hat{\varphi}$:

$$c_0(N,\varphi)=\hat{\varphi}(T)\sum_{r=1}^{r}e^{\pi i m_j/4}e^{-iNS_j}\int_{Y_j}\nu_j,$$

- $Y_j=X_E$ (the flow is periodic);
- $T=kE$ is the (common) period (k the multiplicity);
- $m_j=4$ is the (common) Maslov index of the trajectories in Y_j;
- $S_j=\int_\gamma A+L\sqrt{E^2-1}=k\frac{E^2-1}{2}$ the (common) action.

The two-sphere

Suppose that the manifold M is the round two-sphere

$$x^2 + y^2 + z^2 = R^2.$$

In the spherical coordinates

$$x = R\sin\theta\cos\varphi, y = R\sin\theta\sin\varphi, z = R\cos\theta, \quad \theta \in (0,\pi), \varphi \in (0, 2\pi),$$

the Riemannian metric g on M is given by

$$g = R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

and the magnetic field form F by

$$F = B\sin\theta d\theta \wedge d\varphi.$$

The quantization condition means that

$$B \in \frac{1}{2}\mathbb{Z}.$$

- For $B=\frac{n}{2}, n\in\mathbb{Z}$, the corresponding quantum line bundle L_n can be described as the line bundle associated with the principal S^1-bundle, the Hopf bundle $S^3\to S^2$, and the character $\chi_n: S^1\to S^1$ given by $\chi_n(u)=u^n, u\in S^1$.
- L_n is a well-known Wu-Yang magnetic monopole, which provides a natural topological interpretation of Dirac's monopole of magnetic charge $g=n\hbar/2e$.
- The eigenvalues of the magnetic Laplacian Δ^{L^N} were computed by Wu-Yang (1976) (spherical Landau levels):

$$\nu_{N,j}=\frac{1}{R^2}\left[j(j+1)+\frac{N}{2}(2j+1)\right],\quad j=0,1,\ldots,$$

with multiplicity

$$m_{N,j}=N+2j+1.$$

- The corresponding eigenfunctions are monopole harmonics.

The trace formula

Put $B = \frac{1}{2}$. For any $\varphi \in \mathcal{S}(\mathbb{R})$ and $E > 1$, one has an asymptotic expansion

$$Y_N(\varphi) \sim c_0(N,\varphi)N + \sum_{j=1}^{\infty} c_j(N,\varphi)N^{1-j}, \quad N \to \infty.$$

The coefficients c_j can be computed explicitly.
For the first coefficient, we get

$$c_0(N,\varphi) = \sum_{k\in\mathbb{Z}} 2ER^2 \hat{\varphi}\left(\frac{2\pi ERk}{\sqrt{E^2 - 1 + \frac{1}{4R^2}}}\right) e^{\pi ik(N+1)} e^{-2\pi ikR\sqrt{E^2-1+\frac{1}{4R^2}}N}.$$

The formula simplifies when $R = \frac{1}{2}$.

The hyperbolic plane

Consider the hyperbolic plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ equipped with a Riemannian metric

$$g = \frac{R^2}{y^2}[dx^2 + dy^2].$$

Assume that the Hermitian line bundle $\tilde{L}$ is trivial and the connection $\nabla^{\tilde{L}}$ on $\tilde{L}$ is given by the connection form

$$A = \frac{B}{y}dx.$$

So we have

$$\tilde{F} = B\frac{dx \wedge dy}{y^2}.$$

- $\Gamma \subset PSL(2,\mathbb{R})$ a cocompact lattice acting freely on $\mathbb{H}$.
- $M = \Gamma \setminus \mathbb{H}$ is a compact Riemannian surface of genus $g \geqslant 2$.
- The prequantization condition

$$(2g-2)B \in \mathbb{Z}.$$

- A section of the associated quantum line bundle L on M can be identified with a function ψ on $\mathbb{H}$ such that $g(z) := y^B\psi(z)$ is a Γ-automorphic form of weight $-2B$:

$$g(\gamma z) = (cz+d)^{-2B} g(z), \quad \gamma \in \Gamma, z \in \mathbb{H}.$$

- The corresponding magnetic Laplacian Δ^B first appeared in the theory of automorphic forms (the Maass Laplacian):

$$\Delta^B = -\frac{y^2}{R^2}\left(\left(\frac{\partial}{\partial x} - iBy^{-1}\right)^2 + \frac{\partial^2}{\partial y^2}\right).$$

Its spectrum was computed by Elstrodt (1973).

The trace formula

Put $B = 1$. For any $\varphi \in \mathcal{S}(\mathbb{R})$ and $1 < E < \sqrt{\frac{1}{R^2}+1}$, one has an asymptotic expansion

$$Y_N(\varphi) \sim c_0(N,\varphi)N + \sum_{j=0}^{\infty} c_j(N,\varphi)N^{1-j}, \quad N \to \infty,$$

The coefficients c_j can be computed explicitly.
For the first coefficient, we get

$$c_0(N,\varphi) = (2g-2)ER^2 \sum_{k\in\mathbb{Z}} \hat{\varphi}\left(\frac{2\pi kER}{\sqrt{\frac{1}{R^2}+1-E^2}}\right) e^{ik\pi} e^{2\pi ikR\sqrt{\frac{1}{R^2}+1-E^2}N}.$$

$$c_0(N,\varphi) = (2g-2)ER^2 \sum_{k\in\mathbb{Z}} \hat{\varphi}\left(\frac{2\pi kER}{\sqrt{\frac{1}{R^2}+1-E^2}}\right) e^{ik\pi} e^{2\pi ikR\sqrt{\frac{1}{R^2}+1-E^2}N}.$$

The trace formula is only defined for energies below the Mañé level

$$E < \sqrt{\frac{1}{R^2}+1} \Leftrightarrow \mathcal{E} = \frac{E^2-1}{2} < \mathcal{E}_M = \frac{1}{2R}.$$

Above this energy level, the flow becomes to be Anosov and is conjugate to the geodesic flow on a constant negative curvature surface.

A magnetic analog of the classical Selberg trace formula is a physical interpretation of the classical Maass-Selberg trace formula for the Laplacian on automorphic forms. It doesn't reflect the dynamics of the magnetic flow at fixed energy levels.

The Katok example

$M = S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$.
In the spherical coordinates $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$, the Riemannian metric g on M is given by

$$g = \frac{d\theta^2}{1-\epsilon^2\sin^2\theta} + \frac{\sin^2\theta}{(1-\epsilon^2\sin^2\theta)^2}d\varphi^2, \quad \epsilon \in (0,1),$$

and the magnetic form F by

$$F = dA = \frac{\epsilon\sin 2\theta}{(1-\epsilon^2\sin^2\theta)^2}d\theta\wedge d\varphi, \quad A = -\frac{\epsilon\sin^2\theta}{1-\epsilon^2\sin^2\theta}d\varphi,$$

Since F is exact, the quantum line bundle L is trivial.
In the first order of perturbation theory, i.e. modulo ϵ^2, this system describes the motion of a charged particle on S^2 in the external magnetic field $F = \frac{1}{2}\epsilon\, dx\wedge dy$ parallel to z-axis.

The magnetic geodesic flow

The Hamiltonian H has the form

$$H = \left((1 - \epsilon^2 \sin^2\theta) p_\theta^2 + \frac{(1-\epsilon^2\sin^2\theta)^2}{\sin^2\theta} p_\varphi^2 + 1 \right)^{1/2},$$

and the reduced Hamiltonian system on X_E is given by

$$\begin{aligned}
\dot\theta &= \frac{1}{E}(1 - \epsilon^2 \sin^2\theta) p_\theta, \\
\dot\varphi &= \frac{1}{E}\frac{(1-\epsilon^2\sin^2\theta)^2}{\sin^2\theta} p_\varphi, \\
\dot p_\theta &= \frac{\epsilon^2}{E}\sin\theta\cos\theta p_\theta^2 + \frac{1}{E}\left[\frac{\cos\theta}{\sin^3\theta} - \epsilon^4 \sin\theta\cos\theta\right] p_\varphi^2 + \frac{2\epsilon}{E}\cot\theta p_\varphi, \\
\dot p_\varphi &= -\frac{\epsilon}{E}\frac{\sin 2\theta}{1-\epsilon^2\sin^2\theta} p_\theta.
\end{aligned}$$

The magnetic geodesic flow

This system is integrable with an additional first integral given by

$$P = p_\varphi + \epsilon \frac{\sin^2\theta}{1-\epsilon^2\sin^2\theta}.$$

It is easy to check that this system has two periodic solutions (the equator of the sphere, running in two opposite directions):

$$\theta(t) = \frac{\pi}{2}, \quad \varphi(t) = \pm\frac{\sqrt{E^2-1}}{E}(1-\epsilon^2)t + \varphi_0,$$

$$p_\theta(t) = 0, \quad p_\varphi(t) = \pm\frac{\sqrt{E^2-1}}{1-\epsilon^2}.$$

The magnetic geodesic flow

It turns out that, if $E=\sqrt{2}$ and ϵ is irrational, these are the only periodic orbits of the magnetic geodesic flow. Moreover, they are non-degenerate.
The restriction of the Hamiltonian system to the energy level X_E is described by the Lagrangian

$$L(\theta,\varphi,\dot\theta,\dot\varphi)=\frac{\sqrt{(1-\epsilon^2\sin^2\theta)\dot\theta^2+\sin^2\theta\dot\varphi^2}}{(1-\epsilon^2\sin^2\theta)^2}-\frac{\epsilon\sin^2\theta}{1-\epsilon^2\sin^2\theta}\dot\varphi.$$

As shown by H.-B. Rademacher (2004), this Lagrangian is exactly the Finsler metric introduced by A. Katok (1973).
The Katok example demonstrates that the KAM theorem is false for degenerate Hamiltonians.

Let $\varphi \in \mathcal{S}(\mathbb{R})$, $E = \sqrt{2}$ and $\epsilon \in (0,1)$ is irrational.

The trace formula near zero

If 0 is the only period in $supp(\hat{\varphi})$, then one has an asymptotic expansion

$$Y_N(\varphi) \sim c_0(N,\varphi)N + \sum_{j=1}^{\infty} c_j(N,\varphi)N^{1-j}, \quad N \to \infty,$$

where

$$c_0(N,\varphi) = 2\sqrt{2}\hat{\varphi}(0).$$

The trace formula away zero

If 0 is not in the support of $\hat{\varphi}$, then one has an asymptotic expansion

$$Y_N(\varphi) \sim c_0(N,\varphi) + \sum_{j=1}^{\infty} c_j(N,\varphi)N^{-j}, \quad N \to \infty,$$

The trace formula for non-degenerate periodic orbits

Assume:
there is a unique periodic trajectory $\gamma \subset X_E$ whose period T_γ is in the support of $\hat{\varphi}$, $Y_j = \gamma$ and which is non-degenerate.

$\mathcal{P}_T = \gamma$, $d = 0$ and the density ν on γ can be computed explicitly:

$$c_0(N,\varphi) = \frac{T_\gamma^{\#} e^{\pi i m_\gamma/4}}{2\pi |I - P_\gamma|^{1/2}} e^{-iNS_\gamma} \hat{\varphi}(T_\gamma)$$

- $T_\gamma^{\#}$ the primitive period of γ;
- P_γ the linearized Poincare map of γ;
- m_γ the Maslov index of γ.

γ non-degenerate $\Leftrightarrow$ the map $I - P_\gamma$ is invertible ($|I - P_\gamma| \neq 0$).

The trace formula away zero (continued)

$$c_0(N,\varphi)=\sum_{k\neq 0}\frac{1}{\sqrt{2}(1-\epsilon^2)}\times$$
$$\times\left(\frac{e^{ik\pi m_{k,+}/4}e^{-iNk\frac{2\pi}{1-\epsilon}}}{\sin\frac{\pi k}{1-\epsilon}}+\frac{e^{ik\pi m_{k,-}/4}e^{-iNk\frac{2\pi}{1+\epsilon}}}{\sin\frac{\pi k}{1+\epsilon}}\right)\hat{\varphi}\left(\frac{2\pi\sqrt{2}k}{1-\epsilon^2}\right),$$

$$m_{k,\pm}=2\left[\frac{2k}{1\mp\epsilon}\right]+2\,\mathrm{sign}\,k+1.$$

The trace formula splits into two components, each of them is naturally identified with a contribution of one of equatorial trajectories. The coefficients $c_j^{\pm}(N,\varphi)$ of the asymptotic expansions are spectral invariants of the magnetic Laplacians. They can be considered as analogs of the wave invariants of closed geodesics.

Wave invariants

The study of the wave invariants of closed geodesics was started by Guillemin (1996) and Zelditch (1997, 1998). They are expressed in terms of the coefficients of the quantum Birkhoff normal form for the wave operator in a neighborhood of the closed geodesic, that allows one to compute them in terms of invariants of the Riemannian metric and dynamic invariants of closed geodesics.

Wave invariants of closed magnetic geodesics:

For closed magnetic geodesics, similar invariants haven't been considered yet.

Discussion

- Can one avoid the square roots in the trace formula?
  Recall, for $E > 1$ and $\varphi \in \mathcal{S}(\mathbb{R})$, we define
  $$Y_N(\varphi) = \sum_{j=0}^{\infty} \varphi(\lambda_{N,j} - EN).$$
  where $\lambda_{N,j} = \sqrt{\nu_{N,j} + N^2}$, $\{\nu_{N,j}, j \in \mathbb{N}\}$ the eigenvalues of Δ^{L^N}.
- Maslov type indices (a direct definition, relations with Conley-Zehnder indices (Meinrenken, de Gosson)).
- More explicitly computable (exactly solvable) examples, 3D examples.
- Spectral interpretation of the Mane level.