Classical and quantum semitoric systems

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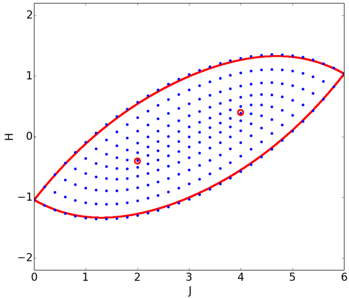
Overview

Goals:

- review semitoric systems: special integrable systems with Hamiltonian S^1-action on four-manifolds,
- introduce their quantum analogues and their joint spectrum,
- explain how this joint spectrum encodes symplectic invariants of the system,
- discuss constructing new examples of such systems.



Section 1

Introduction

Integrable systems

(M^4, ω) (compact) connected symplectic manifold. For $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$:

$$i_{X_f}\omega + df = 0, \qquad \{f, g\} = \omega(X_f, X_g).$$

A (classical) integrable system on M: $f_1, f_2 \in \mathcal{C}^\infty(M, \mathbb{R})$ such that:

- $\{f_1, f_2\} = 0$,
- X_{f_1}, X_{f_2} almost everywhere linearly independent.

$F = (f_1, f_2) : M \to \mathbb{R}^2$ momentum map of the system.

Examples

1. $(M, \omega) = (\mathbb{R}^4, \omega_0)$, $\omega_0 = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$, $f_1 = \xi_1$, $f_2 = \xi_2$,
2. $(M, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\mathbb{S}^2} \oplus \omega_{\mathbb{S}^2})$, $\omega_{\mathbb{S}^2} = dz_i \wedge d\theta_i$:
   - $f_1 = z_1$, $f_2 = z_2$,
   - $f_1 = z_1 + z_2$, $f_2 = x_1x_2 + y_1y_2 + z_1z_2$.

Semiclassical quantization

Roughly: to (M, ω), associate, for $\hbar \to 0$:

1. a Hilbert space $\mathcal{H}_\hbar$ (quantum state space),
2. a linear map $\mathrm{Op}_\hbar : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathscr{S}(\mathcal{H}_\hbar)$ with a number of properties, among which:
   - if $f = c$ constant, then $\mathrm{Op}_\hbar(f) = c\ \mathrm{Id}$,
   - if $f \in L^\infty$, $\|\mathrm{Op}_\hbar(f)\| \sim_{\hbar \to 0} \|f\|$,
   - $[\mathrm{Op}_\hbar(f), \mathrm{Op}_\hbar(g)] = \frac{\hbar}{i} \mathrm{Op}_\hbar(\{f, g\}) + \mathcal{O}(\hbar^2)$.

A semiclassical operator: $T_\hbar = \mathrm{Op}_\hbar(f_\hbar)$ with $f_\hbar = f_0 + \hbar f_1 + \hbar^2 f_2 + \ldots$
$\sigma(T_\hbar) := f_0$ principal symbol.

Two standard contexts:

- $(M, \omega) = (T^*X, d\lambda)$, $X = \mathbb{R}^n$ or (X, g) compact Riemannian manifold: Weyl quantization, pseudodifferential operators, $\mathcal{H}_\hbar = L^2(X)$ (ex.: Schrödinger $-\hbar^2 \Delta + V$),
- (M, ω) compact: geometric quantization, Berezin-Toeplitz operators, $\dim \mathcal{H}_\hbar < +\infty$ (ex.: spin operators).

Quantum integrable systems

A quantum integrable system: a pair $(T_\hbar^{(1)}, T_\hbar^{(2)})$ of semiclassical operators on (M^4, ω) such that:

- $[T_\hbar^{(1)}, T_\hbar^{(2)}] = 0$,
- $(f_1, f_2) = \left(\sigma(T_\hbar^{(1)}), \sigma(T_\hbar^{(2)})\right)$ is a (classical) integrable system.

Joint spectrum $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$: support of joint spectral measure. For instance, if $\dim(\mathcal{H}_\hbar) < \infty$,

$$\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)}) = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \exists v \neq 0,\ T_\hbar^{(1)} v = \lambda_1 v,\ T_\hbar^{(2)} v = \lambda_2 v\right\}.$$

Question

From the data of $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$ when $\hbar \to 0$, which information on (M, ω, f_1, f_2) can we extract?

The toric case

$F = (f_1, f_2)$ integrable on (M^4, ω) is toric if the flows of X_{f_1} and X_{f_2} are 2π-periodic and the action $\mathbb{T}^2 \times M \to M, \ ((t_1, t_2), m) \mapsto (\phi_{f_1}^{t_1} \circ \phi_{f_2}^{t_2})(m)$ is effective.

Theorem (Atiyah, Guillemin-Sternberg 1982, Delzant 1986)

M compact. $P = F(M)$ is a convex polygon such that, at each vertex, the outgoing edges are generated by a $\mathbb{Z}$-basis (u_1, u_2) of $\mathbb{Z}^2$. Moreover, P determines (M, ω, F) up to equivariant symplectomorphism.

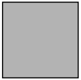


Figure: $\mathbb{S}^2 \times \mathbb{S}^2$.

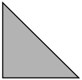


Figure: $\mathbb{CP}^2$.

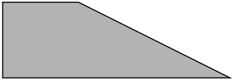


Figure: A Hirzebruch surface.

Theorem (Charles-Pelayo-Vũ Ngọc 2013)

$(T_\hbar^{(1)}, T_\hbar^{(2)})$ quantum integrable system such that $F = (\sigma(T_\hbar^{(1)}), \sigma(T_\hbar^{(2)}))$ toric. From $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$ as $\hbar \to 0$, one can recover (M, ω, F) up to isomorphism.

Idea: $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$ converges towards the Delzant polygon $P = F(M)$ as $\hbar \to 0$.

Section 2

Semitoric systems

Singularities of integrable systems

(f_1, f_2) integrable system on (M^4, ω). Points where X_{f_1}, X_{f_2} linearly dependent: singularities. Notion of nondegenerate singularity to ensure normal form (Eliasson): local coordinates (x_1, x_2, ξ_1, ξ_2) such that $\omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$ and $(f_1, f_2) \sim (q_1, q_2)$ where q_i are some of

1. $q_i = \xi_i$ (regular component),
2. $q_i = \frac{x_i^2 + \xi_i^2}{2}$ (elliptic component),
3. $q_i = x_i \xi_i$ (hyperbolic component),
4. $q_1 = x_1\xi_2 - x_2\xi_1, \; q_2 = x_1\xi_1 + x_2\xi_2$ (focus-focus component).

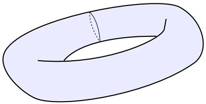


Figure: A regular fiber.

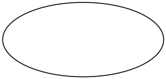


Figure: An elliptic-regular fiber.

Figure: An elliptic-elliptic fiber.

Semitoric systems (Symington, Vũ Ngọc)

An integrable system $F = (J, H) : (M^4, \omega) \to \mathbb{R}^2$ is semitoric if

- J is proper,
- the Hamiltonian flow of J yields an effective $\mathbb{S}^1$-action,
- F has non-degenerate singularities only (like toric case), with no hyperbolic component (these create problems, e.g. disconnected fibers).

New singularities can appear (wrt toric case): focus-focus singularities (in what follows, assume one per J-fiber). Image in the interior of $F(M)$.

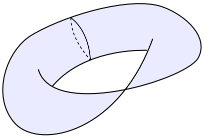


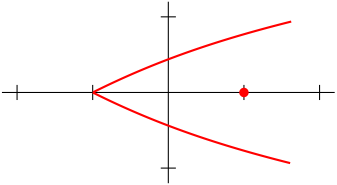
Figure: A focus-focus fiber with one focus-focus point.

Examples

Spin-oscillator

$(M, \omega) = \mathbb{S}^2_{(x,y,z)} \times \mathbb{R}^2_{(u,v)}$, $\omega = \omega_{\mathbb{S}^2} \oplus \omega_{\mathbb{R}^2}$.

$J = \frac{u^2+v^2}{2} + z, \quad H = \frac{ux+vy}{2}$.



Coupled angular momenta (Sadovskií-Zĥilinskií 1999)

$(M, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, R_1\omega_{\mathbb{S}^2} \oplus R_2\omega_{\mathbb{S}^2})$. $R_1, R_2 > 0$. $X = x_1x_2 + y_1y_2$.

$$J = R_1z_1 + R_2z_2, \qquad H_t = (1-t)z_1 + t(X + z_1z_2).$$

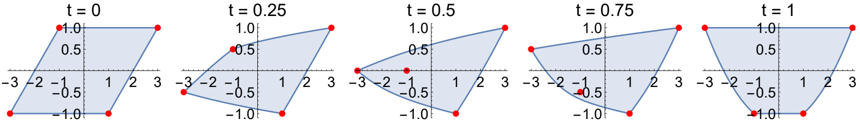


Figure: The momentum map image for varying values of t and $R_1 = 1$, $R_2 = 2$.

Generalization (Hohloch-Palmer 2018)

$(M, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$. $R_1, R_2 > 0$. $X = x_1 x_2 + y_1 y_2$.

$$\begin{cases} J = R_1 z_1 + R_2 z_2, \\ H_{s_1, s_2} = (1 - s_1)(1 - s_2) z_1 + s_1 s_2 z_2 + s_1 (1 - s_2)(X + z_1 z_2) + s_2 (1 - s_1)(X - z_1 z_2), \end{cases}$$

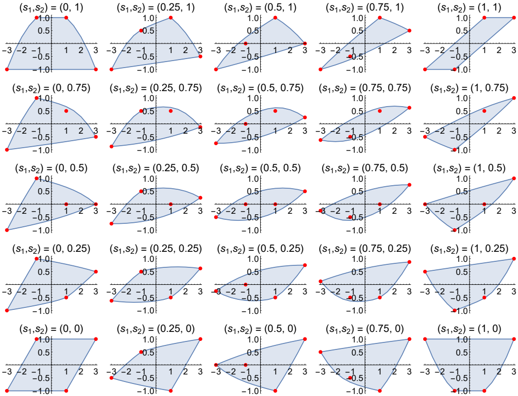


Figure: The momentum map image of the system for $s_1, s_2 \in [0, 1]$ with $R_1 = 1$, $R_2 = 2$.

Classification

(M, ω, F) and (M', ω', F') semitoric are isomorphic $\iff$ $\exists \phi : M \to M'$ symplectomorphism and

$$g : F(M) \to F'(M'), \quad g(x,y) = (x, g^{(2)}(x,y)), \quad \frac{\partial g^{(2)}}{\partial y} > 0$$

such that $F' \circ \phi = g \circ F$.

Theorem (Pelayo-Vũ Ngọc 2007-2011)

Semitoric systems are classified up to isomorphism through "five" invariants:

- *the number m_f of focus-focus singular points,*
- *a family of convex polygons obtained from $F(M)$,*
- *the heights of the images of focus-focus points in these polygons,*
- *a formal series for each focus-focus point,*
- *roughly, an integer for each focus-focus point (twisting index).*

Action variables

$F = (f_1, f_2)$ integrable system on (M^4, ω). Assume regular fibers of F are compact and connected. $c = (c_1, c_2)$ regular value of F.

Theorem (Arnold-Liouville)

$\exists$ neighbourhoods U of $F^{-1}(c)$ in M and V of $\{0\} \times \mathbb{T}^2$ in $\mathbb{R}^2_{(I_1,I_2)} \times \mathbb{T}^2_{(\theta_1,\theta_2)}$ with symplectic form $dI_1 \wedge d\theta_1 + dI_2 \wedge d\theta_2$, a symplectomorphism $\phi : U \to V$ and a diffeomorphism $g : (\mathbb{R}^2, c) \to (\mathbb{R}^2, 0)$ such that $g \circ F \circ \phi^{-1} = (I_1, I_2)$.

- (I_1, I_2): action variables,
- $X_{I_1} = \frac{\partial}{\partial \theta_1}$, $X_{I_2} = \frac{\partial}{\partial \theta_2}$; hence $g \circ F \circ \phi^{-1} : V \to \mathbb{R}^2$ toric momentum map,
- (γ_1, γ_2) basis of $H_1(F^{-1}(c), \mathbb{Z})$ and $d\alpha = \omega$ locally,

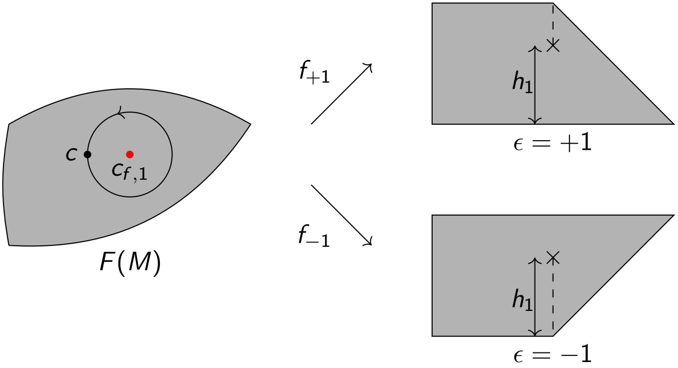
$$g(c) = \frac{1}{2\pi} \left(\int_{\gamma_1(c)} \alpha, \int_{\gamma_2(c)} \alpha \right),$$

- (K_1, K_2) other choice of action variables:

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = A \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}, \quad A \in GL(2, \mathbb{Z}), u, v \in \mathbb{R}.$$

Invariants: semitoric polygons and heights

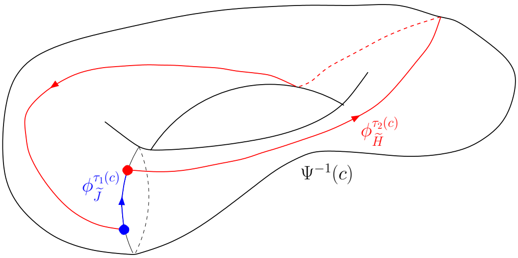
For a semitoric system, $F(M)$ need NOT be a convex polygon.



- Choose independent action variables (J, K) near $F^{-1}(c)$,
- try to extend them: account for monodromy $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,
- action of $\left\{ \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix} \mid k \in \mathbb{Z}, v \in \mathbb{R} \right\}$ on polygons.

Invariants: Taylor series

$\Psi = g \circ F = (\widetilde{J}, \widetilde{H})$, g local diffeo from normal form, $\Psi = (q_1, q_2)$ near the focus-focus point, $q_1 = x_1\xi_2 - x_2\xi_1$, $q_2 = x_1\xi_1 + x_2\xi_2$ in local symplectic coordinates. $c = (c_1, c_2)$ regular, $z = c_1 + ic_2$.



$$\sigma_1(z) = \tau_1(c) - \Im(\mathrm{Log} z), \quad \sigma_2(z) = \tau_2(c) + \Re(\mathrm{Log} z).$$

- $\sigma = \sigma_1 dc_1 + \sigma_2 dc_2$ smooth, closed,
- regularized action S: $dS = \sigma$ near 0, $S(0) = 0$,
- invariant: Taylor series of S at 0.

Invariants: twisting index

- Start with a map f_ϵ from the construction of a semitoric polygon; get generalized toric momentum map $\mu = f_\varepsilon \circ F$,
- from local normal form near focus-focus point m_i, construct local reference toric momentum map ν,
- $\mu = T^{k_i}\nu$, with $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,
- $k_i \in \mathbb{Z}$: twisting index of m_i.

Note: depends on the choice of polygon. Action of $\left(T^p, \begin{pmatrix} 0 \\ v \end{pmatrix} \right)$ changes $k_i \to k_i + p$.

Section 3

The inverse problem for quantum semitoric systems

Question and result

$(T_\hbar^{(1)}, T_\hbar^{(2)})$ semitoric quantum integrable system on (M, ω):

$$F = (J, H) = (\sigma(T_\hbar^{(1)}), \sigma(T_\hbar^{(2)})) \quad \text{semitoric.}$$

Theorem (L.F.-Pelayo-Vũ Ngọc 2016)

From the knowledge of $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$ up to $O(\hbar^2)$, one can recover

- *the number m_f of focus-focus points,*
- *the semitoric polygons of the system,*
- *the height invariant associated with each focus-focus point,*
- *the Taylor series associated with each focus-focus point.*

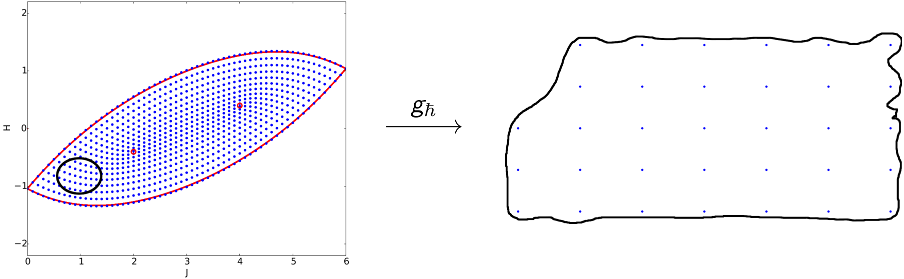
Note: twisting index missing.

Idea of proof 1/2

Bohr-Sommerfeld conditions near c regular value of F (Colin de Verdière 1980, Charbonnel 1988; Charles 2003):

$$(\lambda_1, \lambda_2) \in \mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)}) \iff g_\hbar(\lambda_1, \lambda_2) \in 2\pi\hbar\mathbb{Z}^2 + O(\hbar^\infty);$$

$g_\hbar = g_0 + \hbar g_1 + \ldots$ with $g_0 \circ F = (I_1, I_2)$ action variables:



- reconstruct semitoric polygon,
- find focus-focus values where actions are singular.

Idea of proof 2/2

Taylor series (Pelayo-Vũ Ngọc 2014)

Assume $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)}) = \mathcal{JS}({T'}_\hbar^{(1)}, {T'}_\hbar^{(2)}) + O(\hbar^2)$.

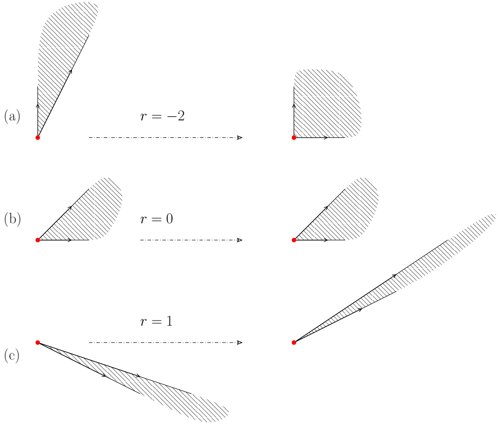
1. Bohr-Sommerfeld conditions $\Rightarrow \exists A \in GL(2,\mathbb{Z})$, $dg_0 = A\ dg_0'$,
2. τ_1, τ_2 as in definition of the Taylor series, $X = \tau_1 X_J + \tau_2 X_H$; compute actions $I = (I_1, I_2)$ using γ_1 orbit of J and γ_2 orbit of X, and similarly I',
3. since g_0 also comes from actions, $\exists B \in GL(2, Z)$ such that $dI = Bdg_0$ and similarly $dI' = Cdg_0'$,
4. this yields $dI = BAC^{-1}dI'$,
5. show that $BAC^{-1} = \mathrm{Id}$, so that $dI = dI'$,
6. one has

$$I_2 = S - \Re(z\mathrm{Log}(z) - z) + K, \quad I_2' = S' - \Re(z\mathrm{Log}(z) - z) + K',$$

where S as in definition of Taylor series invariant, hence $dS = dS'$.

Normalized polygons

Define normalized twisting index from normalized polygon: start from any polygon and apply T^r, $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to get polygon with edge as close as possible to horizontal line.



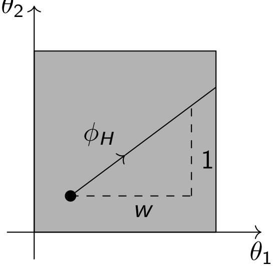
Theorem (L.F.-Pelayo-Vũ Ngọc 2019)

Let $(T_\hbar^{(1)}, T_\hbar^{(2)})$ be a semitoric quantum integrable system with fixed normalized twisting index. From the knowledge of $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$ up to $O(\hbar^2)$, one can recover (M, ω, F) up to isomorphism.

Digression: rotation number

$(T_\hbar^{(1)}, T_\hbar^{(2)})$ quantum integrable system, $F = (f, H) = (\sigma(T_\hbar^{(1)}), \sigma(T_\hbar^{(2)}))$.

- Assume: for every regular value c of F, $F^{-1}(c)$ is compact and connected,
- Arnold-Liouville: near $F^{-1}(c)$, $F = (h_1(I_1, I_2), h_2(I_1, I_2))$, I_1, I_2 action variables, $h = (h_1, h_2)$ diffeomorphism,



Rotation number:

$$w_I(c) = \frac{\frac{\partial h_2}{\partial x}(I_1(m), I_2(m))}{\frac{\partial h_2}{\partial y}(I_1(m), I_2(m))} \in \mathbb{R} \cup \{\infty\},$$

$m \in F^{-1}(c)$.

Theorem (Dauge-Hall-Vũ Ngọc 2019)

From $\mathcal{JS}(T_\hbar^{(1)}, T_\hbar^{(2)})$, recover $w_I(c)$ for every regular value c of F.

Hope to recover the twisting index from the joint spectrum of a semitoric integrable system (but use singular Bohr-Sommerfeld conditions).

Section 4

Semitoric families

General idea

Semitoric systems:

- more invariants than toric ones,
- some of these (polygon, number of focus-focus points) are more visual and easier to detect on joint spectrum,
- constructing a system given its five invariants is complicated and not as explicit as in the toric case,
- few fully explicit examples.

Hence, goal:

- produce more examples,
- systematic construction from given polygon and number of focus-focus points?

Example: coupled angular momenta

$(M, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$. $R_1, R_2 > 0$. $X = x_1 x_2 + y_1 y_2$.

$$J = R_1 z_1 + R_2 z_2, \qquad H_t = (1-t) z_1 + t(X + z_1 z_2).$$

Semitoric with one focus-focus singularity when $t = 1/2$.

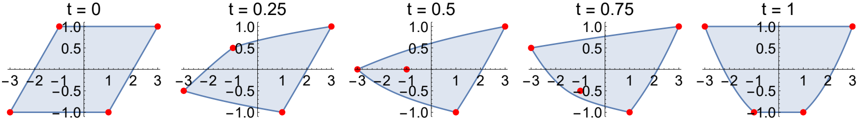


Figure: The momentum map image for varying values of t and $R_1 = 1$, $R_2 = 2$.

Figure: Two representatives of the semitoric polygon of the system when $t = 1/2$.

Idea: generalize this?

Some results

No systematic construction yet, but:

L.F.-Palmer 2019:

- new fully explicit examples on first and second Hirzebruch surfaces,
- use of blowups/blowdowns: examples on every Hirzebruch surface obtained from coupled angular momenta,
- minimal semitoric models (Kane-Palmer-Pelayo) + blowups/blowdowns = everything?

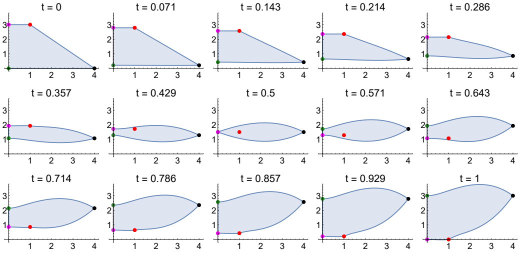


Figure: Image of the momentum map for (J, H_t) on the first Hirzebruch surface.

Thank you!

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Adv. Math., 208(2):909–934.

An example on the second Hirzebruch surface

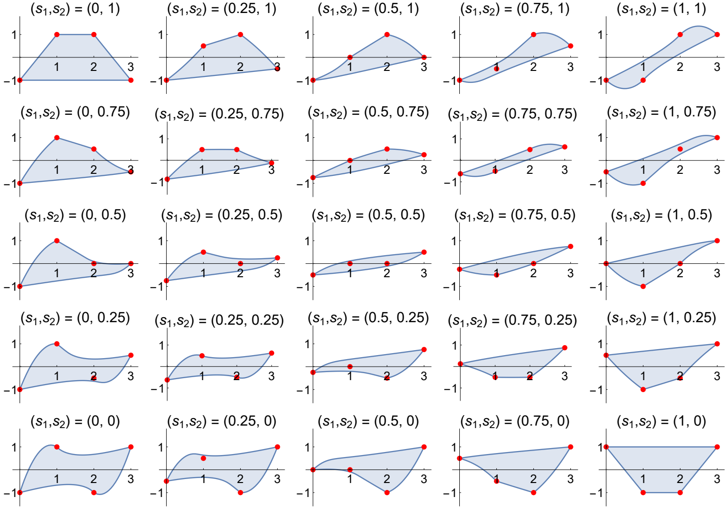

Figure: The image of the momentum map for a system (J, H_{s_1,s_2}) on the second Hirzebruch surface (compare with Hohloch-Palmer).

Interlude: toric blowups

One can perform a blowup at an elliptic-elliptic point of a toric system to get another toric system.

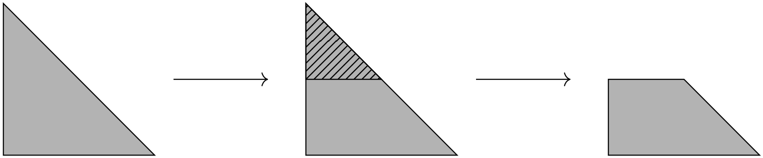

Figure: The momentum map image of $\mathbb{CP}^2$ (left) and a blowup of $\mathbb{CP}^2$ of size $\lambda = 1/2$ (right), obtained by "chopping off" the top corner.

Fulton (1989?): minimal toric models in dimension 4.

Blowups

One can also use blowups and blowdowns at elliptic-elliptic points of a semitoric family to get a new semitoric family. Starting from coupled angular momenta, this yields a semitoric transition family on every Hirzebruch surface:

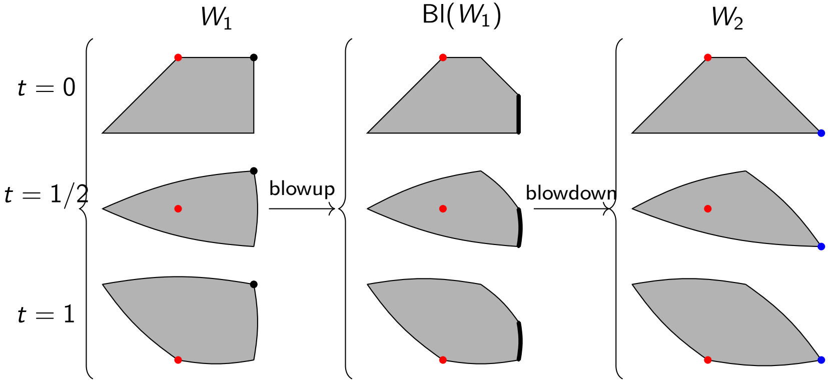

Figure: Performing a blowup followed by a blowdown on the semitoric family on the first Hirzebruch surface to produce a semitoric family on the second Hirzebruch surface. We perform a blowup at the black point and then a blowdown at the bold edge.