Verlinde formulas for nonsimply connected groups

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Workshop on quantization
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Some references:

- Alekseev-M-Woodward: *Formulas of Verlinde type for nonsimply connected groups*. Unfinished manuscript, circa 1998
- Krepski-M, *On the Verlinde formulas for* SO(3)*-bundles*, Quart. J. Math. 64 (2013)
- M: *Verlinde formulas for nonsimply connected groups*, in "Lie groups, geometry and representation theory" – tribute to life and work of Bertram Kostant, Birkhäuser 2018

Kostant's maximal torus in apposition

- G compact, simple, simply connected Lie group
- T maximal torus
- $W = N(T)/T$ Weyl group

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$$w = s_1 \cdots s_l \in W$$

(for some choice of ordered simple roots). The order

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is the Coxeter number of G.

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Basic properties:

1) Coxeter elements form a single conjugacy class in W.
2) The fixed point set of w on T is discrete: $T^w = Z(G)$.

Kostant's maximal torus in apposition

Let h be the Coxeter number.

Theorem (Kostant (1959))

(i) *For all regular elements $g \in G$,*

$$\mathrm{ord}(\mathrm{Ad}(g)) \geq h.$$

Equality holds for a unique conjugacy class $\mathcal{C}_*$ of regular elements.

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(ii) *For $g \in \mathcal{C}_*$, there is a maximal torus U, invariant under* $\mathrm{Ad}(g)$, *such that* $\mathrm{Ad}(g)|_U$ *is a Coxeter transformation.*

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For

$$g = t_* \in \mathcal{C}_* \cap T,$$

one calls the maximal torus U in apposition to T.

Kostant's maximal torus in apposition

Example

$G = \mathrm{SU}(2)$, T=diagonal matrices, $t_* = \mathrm{diag}(i, -i) \in \mathcal{C}_*$,

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$G = \mathrm{SU}(n)$, T=diagonal matrices,

$$t_* = \mathrm{diag}(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{1-n}) \in \mathcal{C}^*$$

where $\zeta = \exp(i\pi/n)$. A maximal torus in apposition is

$$U = \{A \in \mathrm{SU}(n) |\ A_{ij} = A_{i+1,j+1}\}$$

with indices taken mod n.

Kostant's maximal torus in apposition

Conjugacy classes of G are labeled by the Weyl alcove:

$$G/\operatorname{Ad}(G) \cong T/W \cong \mathfrak{A} \subset \mathfrak{t}_+.$$

The conjugacy class $\mathcal{C}_* = \operatorname{Ad}(G)t_*$ corresponds to

$$\xi_* = \frac{1}{h}\rho^\vee.$$

Note: $Z(G)$ acts on $\mathfrak{A}$, fixing ξ_*.

Moduli spaces

- G compact, simply connected, simple Lie group
- Σ compact oriented surface of genus g
- $\mathcal{M}_G(\Sigma)$ moduli space of flat G-connections on Σ
- $\cdot$ basic inner product on $\mathfrak{g}$ $\rightsquigarrow$ symplectic structure on $\mathcal{M}_G(\Sigma)$

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Quasi-Hamiltonian description (Alekseev-Malkin-M, 1997):

$$\mathcal{M}_G(\Sigma) = G^{2\mathrm{g}} /\!\!/ G$$

where G acts by conjugation, with moment map $\Phi\colon G^{2\mathrm{g}} \to G$,

$$\Phi(a_1, b_1, \ldots, a_{\mathrm{g}}, b_{\mathrm{g}}) = \prod a_i b_i a_i^{-1} b_i^{-1}$$

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Unique prequantization at any level $k \in \mathbb{N}$.

Define the level k quantization of $\mathcal{M}_G(\Sigma)$ as index of Dirac operator

$$\mathcal{Q}(\mathcal{M}_G(\Sigma)) = \mathrm{index}(\partial\!\!\!/_L) \in \mathbb{Z};$$

here L is pre-quantum line bundle at level k.

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- $h^\vee$ dual Coxeter number

Moduli spaces

Symplectic Verlinde formula

The level k quantization of $\mathcal{M}_G(\Sigma)$ is given by the formula

$$\mathcal{Q}(\mathcal{M}_G(\Sigma)) = \sum_{\lambda \in \Lambda_k^*} \left(\frac{\# T_{k+\mathsf{h}^\vee}}{|J(t_\lambda)|^2} \right)^{\mathsf{g}-1}$$

where g is the genus of Σ.

Here

- $T_{k+\mathsf{h}^\vee} = \frac{1}{k+\mathsf{h}^\vee}\Lambda^*/\Lambda$ finite subgroup of T
- $t_\lambda = \exp(\frac{\lambda+\rho}{k+\mathsf{h}^\vee}) \in T_{k+\mathsf{h}^\vee}$ 'special elements'

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- $k \in \mathbb{N}$ does not suffice for prequantizability
- Prequantization (if exists) not unique: any two differ by some $\mathrm{Hom}(Z^{2\mathrm{g}}, \mathrm{U}(1))$.

Moduli spaces

Theorem (Krepski, 2006)

The $(G')^{2\mathrm{g}}$ is pre-quantizable at level k $\Leftrightarrow$ k-th multiple of basic inner product takes on integer values on $\Lambda_Z = \exp_T^{-1}(Z)$.

Moduli spaces

Theorem (Krepski, 2006)

The $(G')^{2g}$ is pre-quantizable at level k $\Leftrightarrow$ k-th multiple of basic inner product takes on integer values on $\Lambda_Z = \exp_T^{-1}(Z)$.

New proof based on following fact:

Lemma

A q-Hamiltonian G-space (M, ω, Φ) with abelian fundamental group is pre-quantizable at level k if and only if

1. *$\widetilde{M}$ is pre-quantizable,*
2. *the following invariant*

$$q \colon \pi_1(M) \times \pi_1(M) \to \mathrm{U}(1)$$

is trivial.

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(3) Put

$$q(c_1, c_2) = \exp\Big(2\pi i\, k\Big(\int_{S^1 \times S^1} f^*\omega + \frac{1}{12} \int_{S^1 \times S^1 \times [0,1]} h^*(\theta^L \cdot [\theta^L, \theta^L])\Big)\Big)$$

Moduli spaces

Let $G' = G/Z$ as before; suppose k is such that $(G')^{2\mathrm{g}}$ is prequantized.

To describe $\mathcal{Q}(\mathcal{M}_{G'}(\Sigma)) = \mathrm{index}(\not{\partial}_L)$, note

$$Z(G) \circlearrowright \mathfrak{A} \quad \rightsquigarrow \quad Z(G) \circlearrowright \Lambda_k^* = \Lambda^* \cap k\mathfrak{A}$$

Fuchs-Schweigert

Fuchs-Schweigert formula

$$\mathcal{Q}(\mathcal{M}_{G'}(\Sigma)) = \frac{1}{\# Z^{2\mathrm{g}}} \sum_{\underline{c} \in Z^{2\mathrm{g}}} \epsilon(c_1, \ldots, c_{2\mathrm{g}}) \sum_{\lambda \in \Lambda_k^*,\ c_i \cdot \lambda = \lambda} \left(\frac{\# T_{k+\mathrm{h}^\vee}}{|J(t_\lambda)|^2} \right)^{\mathrm{g}-1}.$$

Here $\epsilon(c_1, \ldots, c_{2\mathrm{g}}) \in \mathrm{U}(1)$ depend on prequantization.

Proved in arXiv:1706.04045 (with precise formula for $\epsilon(\underline{c})$)

Fixed point formula

Starting point: Fixed point formula for level k prequantized q-Hamiltonian spaces with group-valued moment map

$$\Phi\colon M \to G$$

(Alekseev-Woodward-M, 2001).

$$\mathcal{Q}(M/\!\!/G) = \sum_{\lambda\in\Lambda_k^*} \left(\frac{\# T_{k+\mathsf{h}^\vee}}{|J(t_\lambda)|^2}\right)^{-1} \sum_{F\subset M^{t_\lambda}} \zeta_F(t_\lambda)^{1/2} \int_F \frac{\widehat{A}(F)\, e^{\frac{1}{2}c_1(\mathcal{L}_F)}}{D_{\mathbb{R}}(\nu_F, t_\lambda)}.$$

Here $\mathcal{L}_F$ is line bundle for Spin_c-structure on $TM|_F$, and $\zeta_F(t)$ phase factor for action on $\mathcal{L}_F$.

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Computing these phase factors $\zeta_F(t)$ tends to be tricky.

Fixed point formula

Example

For $M = G^{2\mathrm{g}}$, get unique fixed point component of $t = t_\lambda$,

$$F = T^{2\mathrm{g}}.$$

In particular, $\widehat{A}(F) = 1$. Furthermore,

$$D_{\mathbb{R}}(\nu_F, t) = |J(t_\lambda)|^{2\mathrm{g}}, \quad \int_F e^{\frac{1}{2}c_1(\mathcal{L}_F)} = (\# T_{k+\mathrm{h}^\vee})^{\mathrm{g}}, \quad \zeta_F(t)^{1/2} = 1.$$

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$\rightsquigarrow$ Verlinde formulas.

Computation of phase factor was 'easy' since $F \subset M^T$, and $\Phi(F) = \{e\}$.

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Consider fixed point sets for $M = (G')^{2\mathrm{g}}$ with $G' = G/Z$.

Fixed point formula

Consider fixed point sets for $M = (G')^{2g}$ with $G' = G/Z$.

The action $Z(G) \circlearrowright \mathfrak{A}$ defines a group homomorphism

$$Z(G) \to W,\ c \mapsto w_c$$

where

$$c \exp(\xi) = w_c(\exp(\tilde{\xi})),\quad \tilde{\xi} = c.\xi \in \mathfrak{A}.$$

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Lemma

For $t = t_\lambda$,

$$(G')^t = \bigcup_{c \in Z:\ c \cdot \lambda = \lambda} N(T')^{(c)}$$

where $N(T')^{(c)}$ pre-image of w_c.

Fixed point formula

So, for $t = t_\lambda$, $M = (G')^{2\mathrm{g}}$,

$$M^t = \bigcup_{c_1,\ldots,c_{2\mathrm{g}}:\ c_i\cdot\lambda=\lambda} N(T')^{(c_1)} \times \cdots \times N(T')^{(c_{2\mathrm{g}})}$$

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All components $F \subset M^t$ are tori, normal bundle trivial,

$$D_{\mathbb{R}}(\nu_F, t) = |J(t_\lambda)|^{2\mathrm{g}}, \quad \int_F e^{\frac{1}{2}c_1(\mathcal{L}_F)} = \frac{(\# T_{k+\mathrm{h}^\vee})^{\mathrm{g}}}{\# Z^{2\mathrm{g}}}$$

as before.

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as before.

Computation of $\zeta_F(t)^{1/2}$ still tricky.

Fixed point formula

Rescue: Kostant's maximal torus in apposition, U.

Lemma

The maximal torus U meets every $N(T)^{(c)}$, $c \in Z(G)$.

Hence,

$$(U')^{2g} \subset (G')^{2g}$$

meets each component of fixed point set. Note $\Phi((U')^{2g}) = \{e\}$.

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The prequantization of $(G')^{2\mathrm{g}}$ restricts to a prequantization of $(U')^{2\mathrm{g}}$, which we can analyse to figure out $\zeta_F(t)^{1/2}$.

Concluding remarks

- For $G' = \mathrm{PU}(n)$ with n prime, recover formulas of Beauville (1997)
- At some (non-optimal) levels k, the formulas were proved in Alekseev-M-Woodward (1998, unfinished)
- For $G' = \mathsf{SO}(3)$, the formulas were proved in Krepski-M (2012), including cases with boundary.
- For higher rank, case with boundary is much harder (in progress)

Concluding remarks

Thanks.