Quantization, singularities and symmetries in interaction

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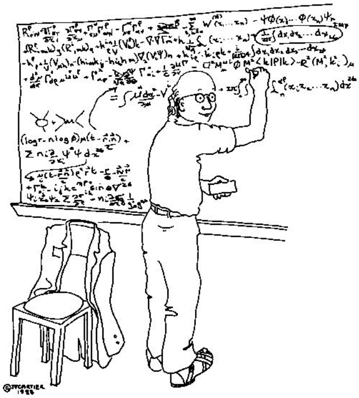
Quantization in Symplectic Geometry

Classical vs. Quantum: The Dream

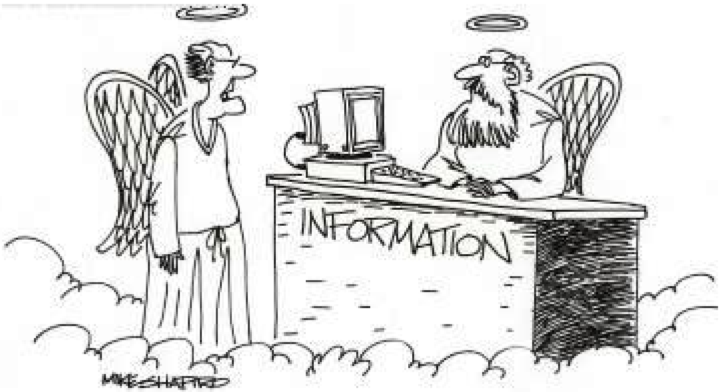
1. Classical systems
2. Observables $C^\infty(M)$
3. Bracket $\{f, g\}$

1. Quantum System
2. Operators in $\mathcal{H}$ (Hilbert)
3. Commutator $[A, B]_h = \frac{2\pi i}{h}(AB - BA)$



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

The truth...



"I still don't understand quantum theory."

Goals

- To revisit several Quantization schemes under the presence of singularities. Singularities can appear in additional data (polarization) or in the symplectic structure itself.
- Claim: Symmetries help! Specially toric, semitoric, almost toric...
- Compare different schemes.

Geometrical Approach I: Geometric Quantization

- (M^{2n}, ω) symplectic manifold with integral $[\omega]$.
- $(\mathbb{L}, \nabla)$ a complex line bundle with a connection ∇ such that $curv(\nabla) = -i\omega$ (prequantum line bundle).
- A real polarization $\mathcal{P}$ is a Lagrangian foliation.
- Integrable systems provide natural examples of real polarizations.
- Flat sections equation: $\nabla_X s = 0$, $\forall X$ tangent to $\mathcal{P}$.

Bohr-Sommerfeld leaves

Definition

A Bohr-Sommerfeld leaf is a leaf of a polarization admitting global flat sections.

Example: Take $M = S^1 \times \mathbb{R}$ with $\omega = dt \wedge d\theta$, $\mathcal{P} =< \frac{\partial}{\partial\theta} >$, $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = td\theta \rightsquigarrow \nabla_X \sigma = X(\sigma) - i < \Theta, X > \sigma$ $\rightsquigarrow$Flat sections: $\sigma(t,\theta) = a(t).e^{it\theta} \rightsquigarrow$ Bohr-Sommerfeld leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.

Liouville-Mineur-Arnold $\leftrightsquigarrow$ this example is the canonical one.

Bohr-Sommerfeld leaves: continued...

Theorem (Guillemin-Sternberg)

If the polarization is a regular fibration with compact leaves over a simply connected base B, then the Bohr-Sommerfeld set is given by,

$$BS = \{p \in M, (f_1(p), \dots, f_n(p)) \in \mathbb{Z}^n\}$$

where $f_1, \dots, f_n$ are global action coordinates on B.

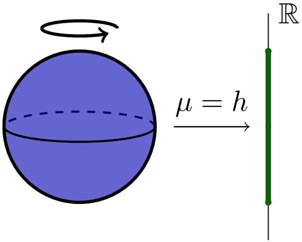
For toric manifolds the base B may be identified with the image of the moment map.

Bohr-Sommerfeld leaves and Delzant polytopes

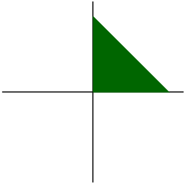
Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes and the bijective correspondence is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



$\mathbb{CP}^2 \xrightarrow{\ \mu\ }$



$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

The case of fibrations

- "Quantize" these systems counting Bohr-Sommerfeld leaves.
- For real polarization given by integrable systems Bohr-Sommerfeld leaves are just "integral" Liouville tori.

Theorem (Sniatycki)

If the leaf space B^n is Hausdorff and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibers, then quantization is given by the count of Bohr-Sommerfeld leaves.

But how exactly?

Quantization: The cohomological approach

- Following the idea of Kostant when there are no global sections we define the quantization of $(M^{2n}, \omega, \mathbb{L}, \nabla, P)$ as

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M, \mathcal{J}).$$

- $\mathcal{J}$ is the sheaf of flat sections.

Then quantization is given by:

Theorem (Sniatycki)

$\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$, *with dimension the number of Bohr-Sommerfeld leaves.*

What is this cohomology?

1. Define the sheaf: $\Omega^i_{\mathcal{P}}(U) = \Gamma(U, \wedge^i \mathcal{P})$..
2. Define $\mathcal{C}$ as the sheaf of complex-valued functions that are locally constant along $\mathcal{P}$. Consider the natural (fine) resolution

$$0 \to \mathcal{C} \xrightarrow{i} \Omega^0_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^2_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \cdots$$

   The differential operator $d_{\mathcal{P}}$ is the one of foliated cohomology.
3. Use this resolution to obtain a fine resolution of $\mathcal{J}$ by twisting the previous resolution with the sheaf $\mathcal{J}$.

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{S} \xrightarrow{\nabla_{\mathcal{P}}} \mathcal{S} \otimes \Omega^1_{\mathcal{P}} \xrightarrow{\nabla_{\mathcal{P}}} \mathcal{S} \otimes \Omega^2_{\mathcal{P}} \to \cdots$$

   with $\mathcal{S}$ the sheaf of sections of the line bundle $\mathbb{L}(\otimes N^{1/2})$.

4. Computation kit: Mayer-Vietoris, Künneth formula, Remarkable fact: S^1-actions help prove semilocal Poincaré lemma (toric, almost toric, semitoric case).

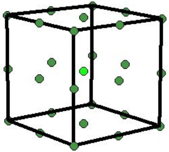
Quantization of toric manifolds

Theorem (Hamilton)

For a $2n$-dimensional compact toric manifold

$$\mathcal{Q}(M) = H^n(M; \mathcal{J}) \cong \bigoplus_{l \in BS_r} \mathbb{C}$$

with a BS_r the set of regular Bohr-Sommerfeld leaves.



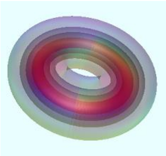
In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

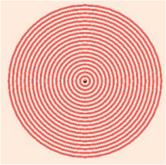
Action-angle coordinates with singularities

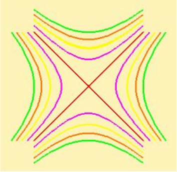
The theorem of Marle-Guillemin-Sternberg for fixed points of toric actions can be generalized to non-degenerate singularities of integrable systems.

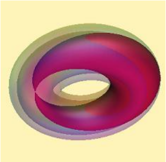
Theorem (Eliasson)

There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.









Liouville torus k_e comp. elliptic k_h hyperbolic k_f focus-focus

Description of singularities

The local model is given by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega = \sum_{i=1}^{k} dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$. and the components of the moment map are:

1. Regular $f_i = p_i$ for $i = 1, ..., k$;
2. Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;
3. Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;
4. focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1$, $j = 1, ..., k_f$.

We say the system is semitoric if there are no hyperbolic components.

The case of surfaces

We can use Čech cohomology computation and a Mayer-Vietoris argument to prove:

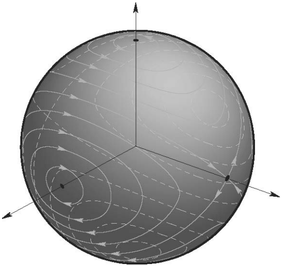
Theorem (Hamilton-M.)

The quantization of a compact surface endowed with an integrable system with non-degenerate singularities is given by,

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l \in BS_r} \mathbb{C} \ ,$$

where $\mathcal{H}$ is the set of hyperbolic singularities.

The rigid body



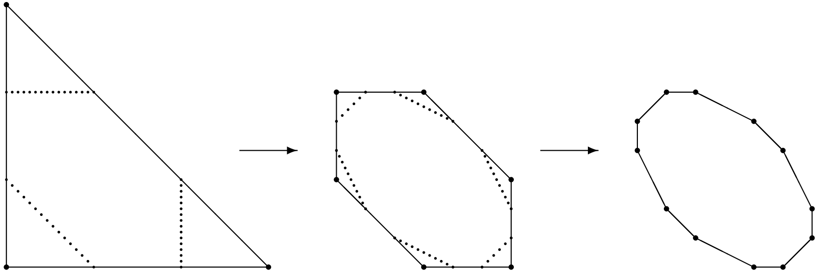
Using this recipe and the quantization of this system is

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}_p^{\mathbb{N}})^2 \oplus \bigoplus_{b \in BS} \mathbb{C}_b.$$

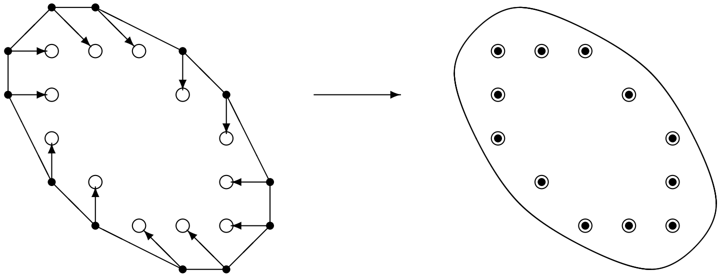
Comparing this system with the one of rotations on the sphere $\rightsquigarrow$ This quantization depends strongly on the polarization.

$\mathbb{C}P^2$, $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$, and $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$

Let us construct toric systems blowing up at 9 singular points using symplectic cutting.

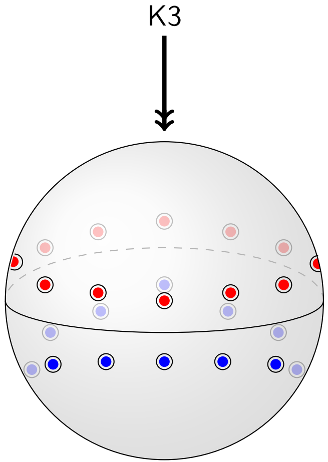


(Symington's) Nodal trades on $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$



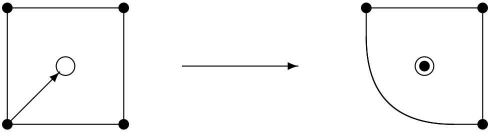
We can convert elliptic points into focus-focus points using nodal trading (Symington).

K3 surface$=(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2)\#_{\mathbb{T}^2}(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2)$



We may glue two copies to obtain a K3 surface.

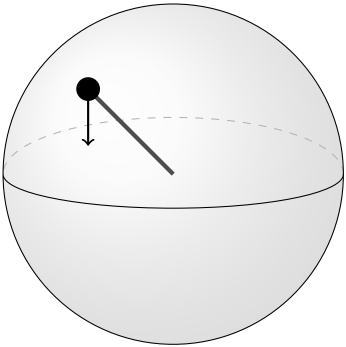
Other examples: Spin-spin system

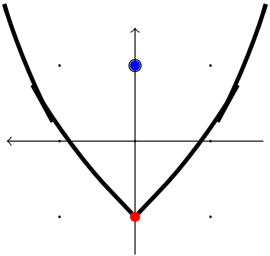


- We may perform a nodal trade on $\mathbb{C}P^1 \times \mathbb{C}P^1$ to obtain a spin-spin system.
- This is a toy model of the spin-spin system of Sadovskií and Zĥilinskií

$$\begin{cases} f_1 = \frac{z_1}{2} + \frac{x_1x_2+y_1y_2+z_1z_2}{2} \\ f_2 = z_1 + z_2 \end{cases}$$

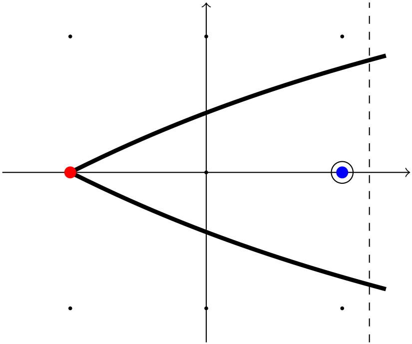
Spherical pendulum





Coupled classical spin and harmonic oscillator $\mathbb{C}P^1 \times \mathbb{C}$

$$\begin{cases} f_1 = z + \frac{1}{2}(u^2 + v^2) \\ f_2 = \frac{1}{2}(xu + yv) \end{cases}$$



Quantization of 4-dimensional almost toric manifolds

Theorem (M-Presas-Solha)

For a 4-dimensional compact almost toric manifold M,

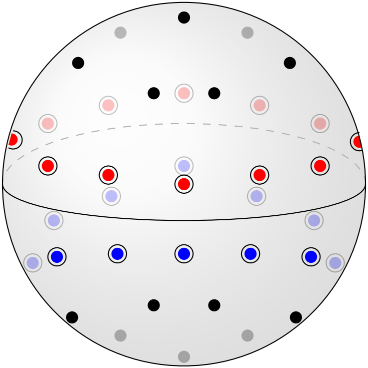
$$\mathcal{Q}(M) \cong \left(\bigoplus_{p \in BS_r} \mathbb{C} \right) \oplus \left(\bigoplus_{p \in BS_f} \oplus_{n(p)} C^{\infty}(\mathbb{R}; \mathbb{C}) \right) ,$$

where with BS_r and BS_f denotes the image of the regular and focus-focus Bohr–Sommerfeld fibers respectively on the base and $n(p)$ the number of nodes on the fiber whose image is $p \in BS_f$.

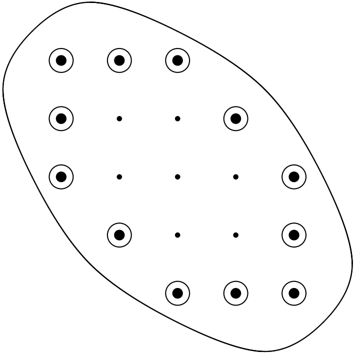
Application: Real geometric quantization of a K3 surface

For a K3 surface with **24** Bohr–Sommerfeld focus-focus fibers;

$$\mathcal{Q}(K3) \cong \mathbb{C}^{14} \oplus \bigoplus_{j\in\{1,\dots,24\}} C^{\infty}(\mathbb{R};\mathbb{C}) \ .$$



Bohr-Sommerfeld leaves in Gompf decomposition of K3



Kähler geometric quantization of a K3 surface

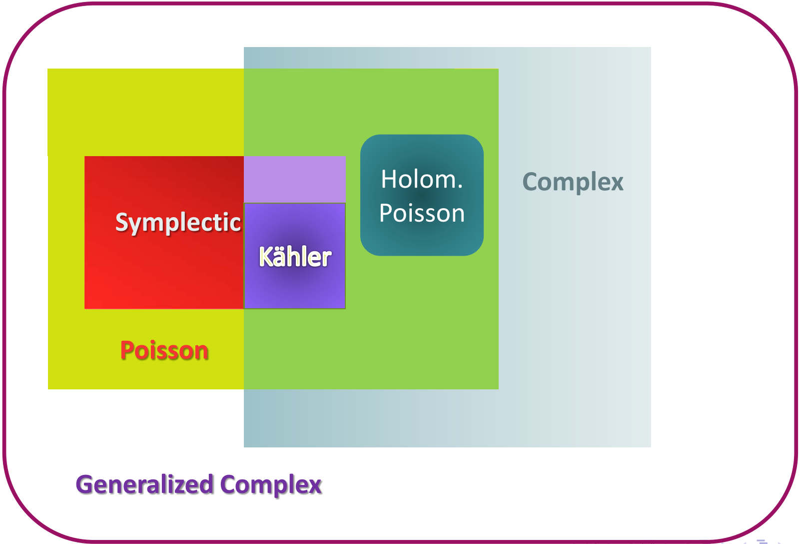
- Dimension of $H^0(K3;L)$ is $\frac{1}{2}c_1(L)^2+2.$ and $c_1(L)^2=\int_{K3}\omega\wedge\omega$
- The symplectic volume of a symplectic sum is the sum of the symplectic volumes $K3=\left(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2\right)\#_{T^2}\left(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2\right)$.
- The symplectic volume of a toric 4-manifold is simply twice the Euclidean volume of its Delzant polytope; thus,

$$\frac{1}{2}c_1(L)^2+2=\frac{1}{2}(2\cdot 24+2\cdot 24)+2=50\ .$$

- and $\mathcal{Q}(K3)\cong\mathbb{C}^{50}$.

Geometric Approach I for singular symplectic manifolds

What do we mean by singular symplectic manifolds?



b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

Theorem

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \ldots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Dimension 2

(Radko) The invariants of b-symplectic structures in dimension 2 are :

- Geometrical: The topology of S and the curves γ_i where Π vanishes.
- Dynamical: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $\lim_{\epsilon\to 0} V_h^\epsilon(\Pi) = \int_{|h|>\epsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$.

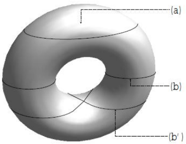
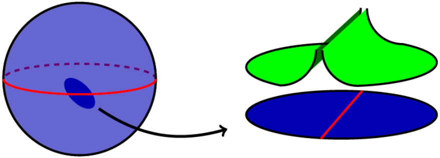


Figure: Two admissible vanishing curves (a) and (b) for Π; the ones in (b') are not admissible.

Examples



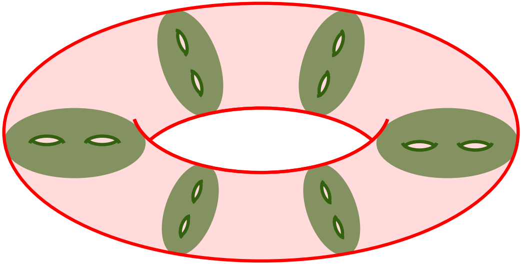
- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
    1. f vanishes linearly.
    2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

This last example is semilocally the *canonical* picture of a b-Poisson structure .

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field v transverse to the symplectic foliation induced on Z.
3. (Guillemin-M. Pires) Z is a mapping torus with glueing diffeomorphism the flow of v.



Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are **b-forms**, ${}^b\Omega^p(M)$.The standard differential extends to

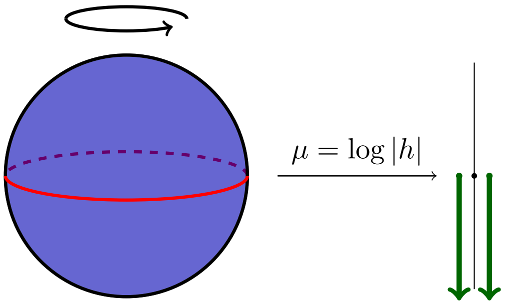
$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

Radko surfaces and their symmetries

$$(S^2, \tfrac{1}{h} dh \wedge d\theta) \leftrightsquigarrow (S^2, h\tfrac{\partial}{\partial h} \wedge \tfrac{\partial}{\partial \theta}).$$

We want to study generalizations of rotations on a sphere.



- Denote by ${}^bC^\infty(M)$ the space of functions which are C^∞ on $M \setminus Z$ and near each Z_i can be written as a sum,

$$c_i \log |f| + g \tag{1}$$

with $c_i \in \mathbb{R}$ and $g \in C^\infty(M)$.

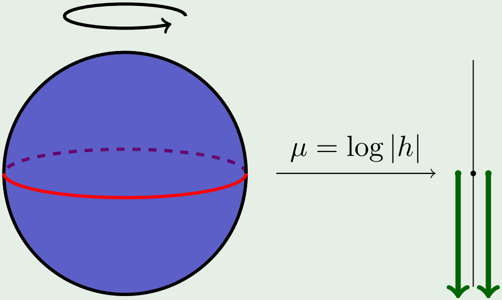
- let T be a torus and $T \times M \to M$ an action of T on M. We will say that this action is *b-Hamiltonian* if the elements, $X \in \mathfrak{t}$ of the Lie algebra of T satisfy

$$\iota(X^M)\,\omega = d\phi, \phi \in {}^bC(M), \tag{2}$$

The S^1-b-sphere

Example

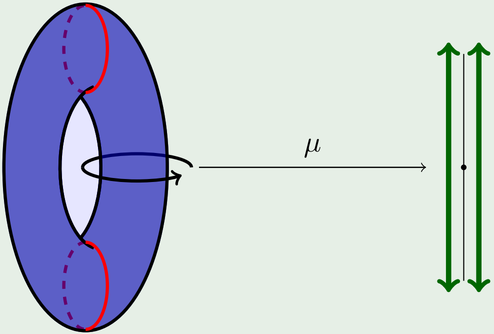
$(\mathbb{S}^2, \omega = \frac{dh}{h} \wedge d\theta)$, with coordinates $h \in [-1,1]$ and $\theta \in [0, 2\pi]$. The critical hypersurface Z is the equator, given by $h = 0$. For the $\mathbb{S}^1$-action by rotations, the moment map is $\mu(h,\theta) = \log|h|$.



The S^1-b-torus

Example

On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface Z is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in θ_2 the moment map is $\mu : \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ is given by $\mu(\theta_1, \theta_2) = \log\left|\frac{1+\cos(\theta_1)}{\sin(\theta_1)}\right|$.

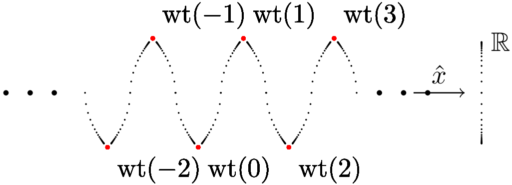


${}^b\mathbb{R}$

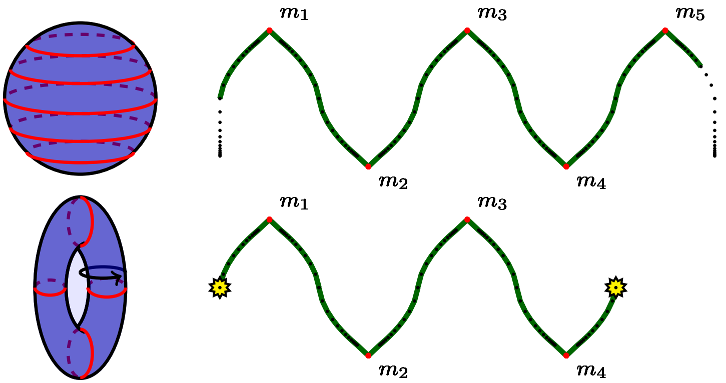
Consider the topological space
${}^b\mathbb{R} \cong (\mathbb{Z} \times \overline{\mathbb{R}})/\{(a, (-1)^a\infty) \sim (a+1, (-1)^a\infty)\}.$ and the local charts $\{\hat{x}\big|_{\{a\}\times\mathbb{R}}, \hat{y}_a\}_{a\in\mathbb{Z}}$ where $\hat{x}(a,x) = x$ and $\hat{y}_a : ((a-1,0),(a,0)) \to \mathbb{R}$,

$$\hat{y}_a = \begin{cases} -\exp\left((-1)^a\hat{x}/\mathrm{w}(a)\right) & \text{in } ((a-1,0),(a-1,(-1)^{a-1}\infty)) \\ 0 & \text{at } (a-1,(-1)^{a-1}\infty) \\ \exp\left((-1)^a\hat{x}/\mathrm{w}(a)\right) & \text{in } ((a,(-1)^{a-1}\infty),(a,0)) \end{cases}.$$

the function $\mathrm{w} : \mathbb{Z} \to \mathbb{R}_{>0}$ associates some weights to the connected components of the critical hypersurface and is determined by the modular periods of each component.



A toric b-surface is defined by a smooth map $f : S \longrightarrow {}^b\mathbb{R}$ or $f : S \longrightarrow {}^b\mathbb{S}^1$ (a posteriori the moment map).



Classification of toric b-surfaces

Theorem (Guillemin, M., Pires, Scott)

A toric b-symplectic surface is equivariantly b-symplectomorphic to either $(\mathbb{S}^2, Z)$ or $(\mathbb{T}^2, Z)$, where Z is a collection of latitude circles.

The action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.

The weights $w(a)$ of the codomain of the moment map are given by the modular periods of the connected components of the critical hypersurface.

The semilocal model

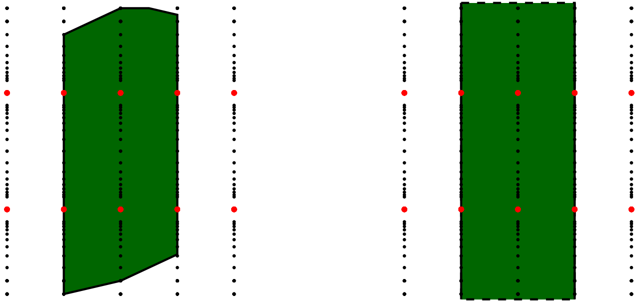
Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$.
For any Delzant polytope $\Delta \subseteq \mathfrak{t}^*_Z$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the b-symplectic manifold as

$$M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\mathrm{lm}} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{\mathrm{lm}}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

From local to global....

We can reconstruct the b-Delzant polytope from the Delzant polytope on a mapping torus via *symplectic cutting* in a neighbourhood of the critical hypersurface.



This information can be recovered by doing reduction in stages: Hamiltonian reduction of an action of $\mathbb{T}_Z^{n-1}$ and the classification of toric b-surfaces.

A b-Delzant theorem

Theorem (Guillemin,M.,Pires, Scott)

The maps that send a b-symplectic toric manifold to the image of its moment map

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*)\} \to \{b\text{-}Delzant\ polytopes\ in\ {}^b\mathfrak{t}^*\} \tag{3}$$

and

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N \rangle)\} \to \{b\text{-}Delzant\ polytopes\ in\ {}^b\mathfrak{t}^*/\langle N \rangle\} \tag{4}$$

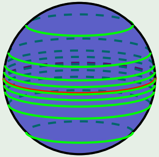
are bijections.

Toric b-manifolds can be of two types either of type ${}^b\mathbb{T}^2 \times X$ (with X a toric symplectic manifold of dimension $(2n-2)$) and manifolds obtained from ${}^b\mathbb{S}^2 \times X$ via symplectic cutting.

Quantization of b-symplectic toric manifolds

Example

Consider on the toric b-sphere: Bohr-Sommerfeld leaves near a connected component of Z in the local model $\omega_\Delta + c\frac{dt}{t} \wedge d\theta$ correspond to $c\log(|h|) = -n$ thus $h = e^{-n/c}$ or $h = -e^{-n/c}$.



Flat sections are given by $s(h,\theta) = f(h)e^{ic\log(|h|)\theta}$ with f analytically flat for $|h| = e^{-n/c}$ and c is the weight of the connected component of Z.

Geometric approach II: Formal Quantization

- Describe a method for quantizing non-compact prequantizable Hamiltonian T-manifolds based upon the "quantization commutes with reduction" principle.
- Important assumption: The moment map ϕ is proper.
- Apply this method to b-symplectic manifolds.

Formal quantization

1. (M, ω) compact symplectic manifold and $(\mathbb{L}, \nabla)$ line bundle with connection of curvature ω.
2. By twisting the spin-$\mathbb{C}$ Dirac operator on M by $\mathbb{L}$ we obtain an elliptic operator $\bar{\partial}_{\mathbb{L}}$.

Since M is compact, $\bar{\partial}_{\mathbb{L}}$ is Fredholm, and we define the geometric quantization $Q(M)$ by

$$Q(M) = \mathrm{ind}(\bar{\partial}_{\mathbb{L}})$$

as a virtual vector space.

Definition of formal quantization (d'après Paradan and Weitsman)

Assume M is non-compact but ϕ proper:
Let $\mathbb{Z}_T \in \mathfrak{t}^*$ be the weight lattice of T and α a regular value of the moment map.
If T acts freely the reduced space $M_\alpha = \phi^{-1}(\alpha)/T$ is a prequantizable symplectic manifold and $[Q, R] = 0$ asserts that $Q(M)_\alpha = Q(M_\alpha)$ where $Q(M)_\alpha$ is the α-weight space of $Q(M)$. We define the formal quantization of M as $Q(M) = \bigoplus_\alpha Q(M_\alpha)$

Theorem (Braverman-Paradan)

$Q(M) = ind(\overline{\partial})$

Formal quantization of b-symplectic manifolds

A b-symplectic manifold is prequantizable if:

- $M \setminus Z$ is prequantizable
- The cohomology classes given under the Mazzeo-Melrose isomorphism applied to $[\omega]$ are integral.

Theorem (Guillemin-M.-Weitsman)

- $Q(M)$ *exists.*
- $Q(M)$ *is finite-dimensional.*

Idea of proof
$Q(M) = Q(M_+) \bigoplus Q(M_-)$
and an ϵ-neighborhood of Z does not contribute to quantization.

Quantum integrable systems on b-symplectic manifolds

Example of classical integrable system on b-symplectic manifolds: b-Toda system standard Toda

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

$$H_2 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + x_1 e^{-q_2} + \sum_{i=2}^{n-1} e^{q_i - q_{i+1}}$$