Horn's problem for pseudo-hermitian matrices

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$$\mathrm{Horn}(G) := \left\{(\mathcal{O}, \mathcal{O}', \mathcal{O}'') \in (\mathfrak{g}/G)^3, \mathcal{O} \subset \mathcal{O}' + \mathcal{O}''\right\}$$

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When G is compact, $\mathrm{Horn}(G)$ is a polyhedron and one can compute the equation of its faces.

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- $\mathfrak{g}_{hol} = G \cdot \mathfrak{t}_{hol}$ where $\mathfrak{t}_{hol} = \mathfrak{g}_{hol} \cap \mathfrak{t}$

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Here $\mathfrak{t}_{hol,+} = \mathfrak{t}_+ \simeq \mathbb{R}_{>0}$ and

$$\mathrm{Horn}_{hol}(SL(2, \mathbb{R})) = \left\{ (a, b, c) \in (\mathbb{R}_{>0})^3, a \geq b + c \right\}$$

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- restriction of π^G_λ to the K-action :

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Proof of Theorem A

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- Kirwan polyhedron $\Delta_{b,c} = \Phi_{b,c}(\mathcal{X}) \cap \mathfrak{t}^*_+$

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Another model

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$$\mathrm{Horn}_{hol}(G) := \left\{(a, b, c) \in (\mathfrak{t}^*_{hol,+})^3, \;\; a \in \Delta_{b,c}\right\}$$

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$M =$ complex K-manifold with finite generic stabilizers

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(γ, C) is a Ressayre pair if $\exists m \in C$ such that

$$\mathfrak{k}_m = \mathbb{R}\gamma \qquad \text{and} \qquad \mathfrak{n}^{\gamma<0} \simeq (\mathrm{T}_m M)^{\gamma<0}$$

Kirwan polyhedron & Ressayre pair

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An element $\xi \in \mathfrak{t}^*_+$ belongs to Δ_d if and only if

$$\langle \xi, \gamma \rangle \leq \langle \Phi_d(C), \gamma \rangle$$

for any Ressayre pair (γ, C).

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$(a,b,c) \in \mathrm{Horn}(n)$ if $|a| = |b| + |c|$ and for any $1 \leq r \leq n-1$, for any $I, J, L \subset \{1,\ldots,n\}$ of cardinal r, we have

$$|a|_I \leq |b|_J + |c|_L$$

if $(\mu(I), \mu(J), \mu(L)) \in \mathrm{Horn}(r)$.

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- $$|a|_I \leq |b|_J + |c|_L$$
  if $(\mu(I), \mu(J), \mu(L)) + (n - r)(\delta, \delta, \delta) \in \mathrm{Horn}(r, 1)$.