Spectral analysis of sub-Riemannian Laplacians, Weyl measures

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Sub-Riemannian Laplacian

(M, D, g) sub-Riemannian (sR) structure:

- M smooth connected manifold of dimension n (may have a boundary)
- $m \in \mathbb{N}^*$, $D = \mathrm{Span}(X_1, \dots, X_m) \subset TM$ (horizontal distribution: subsheaf)
- sR metric defined by

$$\forall q \in M \qquad \forall v \in D_q \qquad g_q(v, v) = \inf \left\{ \sum_{i=1}^m u_i^2 \;\middle|\; v = \sum_{i=1}^m u_i X_i(q) \right\}$$

μ: arbitrary smooth volume form on M

$$\boxed{\triangle = -\sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m \left(X_i^2 + \mathrm{div}_\mu(X_i) X_i \right)}$$

(X_i^*: adjoint in $L^2(M, \mu)$)

Sub-Riemannian Laplacian

Equivalent definitions:

- $-\triangle$ = selfadjoint nonnegative operator on $L^2(M,\mu)$ defined as the Friedrichs extension of the Dirichlet integral

$$Q(\phi) = \int_M \|d\phi\|_{g^*}^2 \, d\mu \qquad \phi \in C_c^\infty(M)$$

$$\left(g^*(\xi,\xi) = \max_{v\in D_q\setminus\{0\}} \frac{\langle \xi, v\rangle^2}{g_q(v,v)} \text{ cometric associated with } g \right)$$

- $\boxed{\triangle\phi = \mathrm{div}_\mu\left(\nabla_{sR}\phi\right)}$ where:

div_μ defined by $L_X\, d\mu = \mathrm{div}_\mu(X)\, d\mu \quad \forall X$ vector field on M

∇_{sR} horizontal gradient defined by $g_q(\nabla_{sR}\phi(q), v) = d\phi(q).v \quad \forall v \in D_q$

note that $\|d\phi\|_{g^*} = \|\nabla_{sR}\phi\|_g$

Hörmander operators

More generally:

X_0 smooth vector field on M, c smooth function on M that is bounded above

$$\boxed{\triangle = \sum_{i=1}^{m} X_i^2 + X_0 + c\,\mathrm{id}}$$

$\rightarrow$ operator on $L^2(M, \mu)$

Remark: $\triangle$ symmetric $\Leftrightarrow X_0 = \sum_{i=1}^{m} \mathrm{div}_\mu(X_i) X_i$

$e(t) =$ Schwartz kernel of $e^{t\triangle}$, of density $e_{\triangle,\mu}(t)$ w.r.t. μ.

$\rightarrow$ heat kernel $e_{\triangle,\mu} : (0, +\infty) \times M \times M \rightarrow (0, +\infty)$

Objective

- Establish small-time asymptotics for the heat kernel.

Spectral properties of sR Laplacians

$$\boxed{\triangle = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} \left(X_i^2 + \mathrm{div}_\mu(X_i) X_i \right)}$$

For M compact, under Hörmander's assumption $\mathrm{Lie}(D) = TM$, the operator $-\triangle$ is hypoelliptic (and even subelliptic), has a compact resolvent and thus a discrete spectrum

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \to +\infty$$

Let $(\phi_k)_{k \in \mathbb{N}^*}$ be an orthonormal eigenbasis of $L^2(M, \mu)$.

Objectives

- Derive (micro-)local Weyl laws for $\triangle$, identify (micro-)local Weyl measures.
- Establish Quantum Ergodicity (QE) properties.

sR flag

Sequence of subsheafs $D^k \subset TM$:

- $D^0 = \{0\}$
- $D^1 = D = \mathrm{Span}(X_1, \ldots, X_m)$
- $D^{k+1} = D^k + [D, D^k]$ for $k \geqslant 1$

sR flag at q: $\boxed{\{0\} = D_q^0 \subset D_q = D_q^1 \subset D_q^2 \subset \ldots \subset D_q^{r(q)-1} \subsetneq D_q^{r(q)} = T_qM}$

$r(q)$: degree of nonholonomy at q

- $n_i(q) = \dim D_q^i \qquad (n_{r(q)}(q) = n = \dim M)$
- $w_1(q) = \cdots = w_{n_1}(q) = 1$
  $w_{n_1+1}(q) = \cdots = w_{n_2}(q) = 2$
  $\vdots$
  $w_{n_{r-1}+1}(q) = \cdots = w_{n_r}(q) = r$

$$\mathcal{Q}(q) = \sum_{i=1}^{r} i(n_i(q) - n_{i-1}(q)) = \sum_{i=1}^{n} w_i(q)$$

(= Hausdorff dimension around q if q regular)

q is said regular if the flag at q is regular.
The sR structure is said equiregular if all points are regular.

Nilpotentization

$(\widehat{M}^{q_0}, \widehat{D}^{q_0}, \widehat{g}^{q_0})$ = nilpotentization of the sR structure (M, D, g) at $q_0 \in M$
= tangent space (in the metric sense of Gromov)
(equivalence class under the action of sR isometries on sR structures)

- $\widehat{M}^{q_0} \sim \mathbb{R}^n$
- $\widehat{D}^{q_0} = \mathrm{Span}(\widehat{X}_1^{q_0}, \dots, \widehat{X}_m^{q_0})$

In privileged coordinates:

- Dilation $\delta_\varepsilon(x) = \left(\varepsilon^{w_1(q_0)} x_1, \dots, \varepsilon^{w_n(q_0)} x_n\right)$
- $\widehat{X}_i^{q_0} = \lim_{\varepsilon \to 0} \varepsilon \delta_\varepsilon^* X_i$
- $\widehat{\mu}^{q_0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\mathcal{Q}(q_0)}} \delta_\varepsilon^* \mu = \mathrm{Cst}(q_0)\, dx$

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Nilpotentized sR Laplacian: $\boxed{\widehat{\triangle}^{q_0} = \sum_{i=1}^{m} \left(\widehat{X}_i^{q_0}\right)^2}$

Heat kernel: $e_{\widehat{\triangle}^{q_0}, \widehat{\mu}^{q_0}} : (0, +\infty) \times \widehat{M}^{q_0} \times \widehat{M}^{q_0} \to \mathbb{R}$

Remark: Homogeneity $e_{\widehat{\triangle}^{q_0}, \widehat{\mu}^{q_0}}(t, x, x') = \varepsilon^{\mathcal{Q}(q_0)}\, e_{\widehat{\triangle}^{q_0}, \widehat{\mu}^{q_0}}(\varepsilon^2 t, \delta_\varepsilon(x), \delta_\varepsilon(x')) \quad \forall \varepsilon \in \mathbb{R}$

Heat kernel asymptotics

Fundamendal lemma

$q_0 \in M$ arbitrary, μ arbitrary smooth measure on M. In a local chart of privileged coordinates at q_0:

$$\forall k \in \mathbb{N} \qquad t^{\mathcal{Q}(q_0)/2}\, e_{\triangle,\mu}\left(t, \delta_{\sqrt{t}}(x), \delta_{\sqrt{t}}(x')\right) = e_{\widehat{\triangle}^{q_0},\widehat{\mu}^{q_0}}(1,x,x') + t\, a_1(x,x') + \cdots + t^k\, a_k(x,x') + \mathrm{o}(t^k)$$

as $t \to 0^+$, in $C^\infty(M \times M)$ topology, with a_j smooth.

- If q_0 is regular, then the above convergence and asymptotic expansion are locally uniform with respect to q_0.
- Still valid for $\triangle = \sum_{i=1}^m X_i^2 + X_0 + c\,\mathrm{id}$, with an expansion in $t^{k/2}$, provided that:
  - either X_0 smooth section of D;
  - or X_0 smooth section of D^2, and then replace $\widehat{\triangle}^{q_0}$ with $\widehat{\triangle}^{q_0} + \widehat{X}_0^{q_0}$.

Heat kernel asymptotics

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as $t \to 0^+$, in $C^\infty(M \times M)$ topology, with a_j smooth.

- $x = x' = 0 \Rightarrow$ expansion of the kernel along the diagonal, and

$$e_{\triangle,\mu}(t,q_0,q_0)\, d\mu(q_0) \sim \frac{e_{\widehat{\triangle}^{q_0},\widehat{\mu}^{q_0}}(1,0,0)}{t^{\mathcal{Q}(q_0)/2}}\, d\mu(q_0) = e_{\widehat{\triangle}^{q_0},\widehat{\mu}^{q_0}}(t,0,0)\, d\mu(q_0)$$

  $\to$ useful to derive the local Weyl law.
  Generalization of results by Métivier (1976), Ben Arous (1989).

- estimations **near** the diagonal $\to$ microlocal Weyl law and singular sR structures.

General question in sR geometry: define an intrinsic volume on a sR manifold.

[Agrachev Barilari Boscain 2012]

- Hausdorff (standard or spherical) Mitchell 1985, Gromov 1996
- Popp Montgomery 2002
- a new one: Weyl

Popp measure

Popp volume (Montgomery 2002): canonical smooth volume form defined at q regular by

$$|dP(q)| = \Phi^* |\nu_1 \wedge \cdots \wedge \nu_r|$$

with the canonical isomorphism
$$\Lambda^n(T_q^\star M) \xrightarrow{\Phi} \Lambda^n\left(\bigoplus_{k=1}^{r(q)} D_q^k / D_q^{k-1}\right)^\star$$

and with $\nu_k =$ canonical volume form on D_q^k/D_q^{k-1} induced by the Euclidean structure coming from the surjection $D_q^{\otimes k} \to D_q^k / D_q^{k-1}$ (take Lie brackets modulo D_q).

- P is invariant under (local) sR isometries
- P commutes with nilpotentization: $\widehat{P_M}^q = P_{\widehat{M}^q}$

$\Rightarrow$ P is “doubly intrinsic”

Explicit expression in [Barilari Rizzi 2013]

(Micro-)local Weyl measure

M compact

Local Weyl measure = probability measure $w_\triangle$ on M defined (if the limit exists) by

$$\int_M f\, dw_\triangle = \lim_{t\to 0+} \frac{\mathrm{Tr}\left(f\, e^{t\triangle}\right)}{\mathrm{Tr}\left(e^{t\triangle}\right)} = \lim_{t\to 0+} \frac{\int_M e(t,q,q) f(q)\, d\mu(q)}{\int_M e(t,q,q)\, d\mu(q)} \qquad \forall f \in C^0(M)$$

i.e.,

$$\boxed{w_\triangle = \text{weak} \lim_{t\to 0+} \frac{e(t,q,q)}{\int_M e(t,q',q')\, d\mu(q')}\, \mu}$$

Microlocal Weyl measure = probability measure $W_\triangle$ on $S^\star M$ defined (if the limit exists) by

$$\int_{S^\star M} a\, dW_\triangle = \lim_{t\to 0+} \frac{\mathrm{Tr}\left(\mathrm{Op}(a) e^{t\triangle}\right)}{\mathrm{Tr}\left(e^{t\triangle}\right)} \qquad \forall a \in \mathcal{S}^0(S^\star M)$$

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- They do not depend on μ, nor on the quantization.
- $\pi_* W_\triangle = w_\triangle$ with $\pi : T^*M \to M$ canonical projection.
- $w_\triangle$ is invariant under sR isometries of M.
- In the equiregular case: $w_\triangle$ exists, is smooth (cf further) and commutes with nilpotentization: $\widehat{w_\triangle}^q =$ "$w_{\widehat{\triangle}^q}$"
- $\mathrm{supp}(W_\triangle) \subset S\Sigma$, where $\Sigma = D^\perp$ (annihilator of D).

(Micro-)local Weyl measure

Equivalent definition (by the Karamata tauberian theorem):

$-\triangle\phi_k = \lambda_k\phi_k$, $\quad (\phi_k)_{k\in\mathbb{N}^*}$ orthonormal eigenbasis of $L^2(M,\mu)$, $\quad \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \to +\infty$

Spectral counting function: $\quad N(\lambda) = \#\{k \mid \lambda_k \leqslant \lambda\}$

Local Weyl measure = probability measure $w_\triangle$ on M defined (if the limit exists) by

$$\int_M f\,dw_\triangle = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k\leqslant\lambda} \int_M f|\phi_k|^2\,d\mu \qquad \forall f\in C^0(M)$$

i.e.,

$$\boxed{w_\triangle = \text{weak} \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k\leqslant\lambda} |\phi_k|^2\,\mu} \qquad \text{(Cesàro mean)}$$

Microlocal Weyl measure = probability measure $W_\triangle$ on $S^\star M$ defined (if the limit exists) by

$$\int_{S^\star M} a\,dW_\triangle = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k\leqslant\lambda} \langle \mathrm{Op}(a)\phi_k, \phi_k\rangle_{L^2(M,\mu)} \qquad \forall a\in\mathcal{S}^0(S^\star M)$$

(Micro-)local Weyl measure

Defining the *averaged correlation* of eigenfunctions by

$$C(x,y) = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k\leqslant\lambda} \phi_k(x+y/2)\overline{\phi_k(x-y/2)} \qquad \forall (x,y)\in M\times M$$

(when the limit exists), the correlation is the Fourier transform with respect to ξ of $W_\triangle$, i.e.,

$$\boxed{C(x,y) = \int e^{-iy.\xi}\, dW_\triangle(x,\xi)}$$

(Micro-)local Weyl measure

Objective

Identify the Weyl measures (equiregular and singular cases).

Compare them with Hausdorff, Popp.

Local Weyl law in the equiregular case

Theorem

In the equiregular case, the local Weyl measure $w_\triangle$ exists, is smooth, and

$$\boxed{dw_\triangle(q) = \frac{\widehat{e}^q(1,0,0)}{\int_M \widehat{e}^{q'}(1,0,0)\,dP(q')}\,dP(q)} \qquad (\text{with } \widehat{e}^q = e_{\widehat{\triangle}^q,\widehat{P}^q})$$

<u>Proof:</u> Along the diagonal, $t^{\mathcal{Q}/2}e_{\triangle,\mu}(t,q,q)\,d\mu(q) \longrightarrow \widehat{e}^q(1,0,0)\,dP(q)$ as $t \to 0^+$.

<u>Remark:</u> Since $w_\triangle$ is smooth, it differs in general from $\mathcal{H}_S$ (which is not smooth in general for $n \geqslant 5$, see [Agrachev Barilari Boscain 2012])

<u>Consequence:</u> $$\boxed{N(\lambda) \sim \frac{\int_M \widehat{e}^q(1,0,0)\,dP(q)}{\Gamma(\mathcal{Q}/2+1)}\lambda^{\mathcal{Q}/2}} \quad \text{as } \lambda \to +\infty \quad (\mathcal{Q}\text{: Hausdorff dim})$$

Example: 3D contact case, $N(\lambda) \sim \frac{1}{32}\lambda^2$

Local Weyl law in the equiregular case

$w_\triangle = P$ (i.e., Weyl = Popp) $\iff$ $q \mapsto \widehat{e}^q(1,0,0)$ is constant

Remark:

$\mathrm{Iso}_{sR}(M)$ acts transitively on M

$\Rightarrow$ all nilpotent sR structures $(\widehat{M}^q, \widehat{D}^q, \widehat{g}^q)$ are sR-isometric

$\Rightarrow$ $q \mapsto \widehat{e}^q(1,0,0)$ is constant

This is so in the following cases:

- The sR structure on M is free
- The sR structure on M is nilpotent and equiregular, and $\dim M \leqslant 5$, with:
  - dimension 3: growth vector (2, 3) (Heisenberg)
  - dimension 4: growth vector (2, 3, 4) (Engel) and (3, 4) (quasi-Heisenberg)
  - dimension 5: growth vector (2, 3, 5) (Cartan), (2, 3, 4, 5) (Goursat rank 2), (3, 5) (corank 2), (3, 4, 5) (Goursat rank 3)

[Agrachev Barilari Boscain 2012]

Remark: $w_\triangle \neq P$ in general (example: bi-Heisenberg $(4,5)$ case)

Microlocal Weyl law in the equiregular case

Define $\Sigma^i = (D^i)^\perp \subset T^\star M$ (annihilator of D^i) for $i = 1, \ldots, r$.

Theorem

In the equiregular case, the microlocal Weyl measure $W_\triangle$ exists, and

$$dW_\triangle(q,u) = \frac{1}{(2\pi)^n}\left(\int_0^{+\infty}\int_{T_q^*M/\Sigma_q^{r-1}} K(q,P_1,ru)\,dP_1\,r^{n-n_{r-1}-1}\,dr\right)\,d\mathcal{H}_{S\Sigma^{r-1}}(q,u)$$

with $K(q,p) = \int_{\mathbb{R}^n} e^{-iy.p} e_{\widehat{\triangle}^q,\widehat{\mu}^q}(1,y,0)\,d\widehat{\mu}^q(y) = \mathcal{F}\Big(y \mapsto e_{\widehat{\triangle}^q,\widehat{\mu}^q}(1,y,0)\Big)(p)$

In particular:

$$\boxed{\mathrm{supp}(W_\triangle) \subset S\Sigma^{r-1}}$$

Two simple singular sR structures

Regular Grushin case:

Almost-Riemannian structure on M compact 2D, Riemannian except along a 1D submanifold $\mathcal{S}$, with no tangency points.

- Local model: $X = \partial_x$, $Y = x\partial_y$, $\quad \mathcal{S} = \{x = 0\}$.
- In $M \setminus \mathcal{S}$, we have $dP = \frac{1}{|x|} dx\, dy = \frac{1}{|x|} dx \otimes d\nu$ with ν smooth.

Regular Martinet case:

Rank-two sR structure on M compact 3D, with $D = \ker \alpha$, with $\alpha \wedge d\alpha$ vanishing on $\mathcal{S}$ (Martinet surface), D transverse to $\mathcal{S}$.

$$\Leftrightarrow D^2 \neq D^3, \quad D^2 = TM \text{ outside of } \mathcal{S}, \quad D^3 = TM \text{ along } \mathcal{S}, \quad D \cap T\mathcal{S} \text{ line bundle over } \mathcal{S}.$$

- Local model: $\alpha = dz - \frac{x^2}{2} dy \quad \left(X = \partial_x,\ Y = \partial_y + \frac{x^2}{2}\partial_z\right), \quad \mathcal{S} = \{x = 0\}$.
- In $M \setminus \mathcal{S}$, we have $dP = \frac{1}{x^2} dx\, dy\, dz = \frac{1}{x^2} dx \otimes d\nu$ with ν smooth.

Two simple singular sR structures

Small-time expansion along the diagonal at any order:

- Grushin:

$$\int_M e_{\triangle,\mu}(t,q,q)f(q)\,d\mu = a_1\frac{|\ln t|}{t} + a_2\frac{1}{t} + a_3\frac{|\ln t|}{\sqrt{t}} + a_4\frac{1}{\sqrt{t}} + a_5|\ln t| + a_6 + a_7\sqrt{t}|\ln t| + a_8\sqrt{t} + a_9 t|\ln t| + a_{10}t + \cdots$$

with $a_1 = \frac{1}{4\pi}\int_{\mathcal{S}} f\,d\nu$ and $a_2 = \frac{1}{4\pi}(\gamma + 4\ln 2)\int_{\mathcal{S}} f\,d\nu + \mathrm{P.V.}\int_M f\,dx_g$

- Martinet:

$$\int_M e_{\triangle,\mu}(t,q,q)f(q)\,d\mu = a_1\frac{|\ln t|}{t^2} + a_2\frac{1}{t^2} + a_3\frac{|\ln t|}{t^{3/2}} + a_4\frac{1}{t^{3/2}} + a_5\frac{|\ln t|}{t} + a_6\frac{1}{t} + a_7\frac{|\ln t|}{\sqrt{t}} + a_8\frac{1}{\sqrt{t}} + a_9|\ln t| + a_{10} + \cdots$$

with $a_1 = \frac{1}{16}\int_{\mathcal{S}} f\,d\nu$

Consequence: $w_\triangle = \frac{\nu}{\nu(\mathcal{S})}$ $W_\triangle = \frac{1}{2}\pi^*_{|\Sigma} w_\triangle$ ($\pi_{|\Sigma} : S\Sigma \to M$ double covering)

Grushin : $N(\lambda) \sim \frac{\nu(\mathcal{S})}{4\pi}\lambda\ln\lambda$ Martinet : $N(\lambda) \sim \frac{\nu(\mathcal{S})}{32}\lambda^2\ln\lambda$

$\Rightarrow$ spectral concentration on the singular manifold $\mathcal{S}$

Singular sR structures

$\mathcal{S}$: singular set of the sR structure

Theorem

If $\mathcal{S}$ is Whitney stratifiable then there exists a submanifold N of M s.t.

$$\int_M e_{\triangle,\mu}(t,q,q) f(q)\, d\mu(q) \underset{t\to 0^+}{\sim} \mathrm{Cst}\,\frac{|\ln^k t|}{t^r}\int_N f\, dw_\triangle \qquad \forall f\in C^0(M)$$

(concentration on N)

for some $k\in\{0,1,\dots,n\}$ and $r\in\mathbb{Q}$ s.t. $r\geqslant \frac{\mathcal{Q}_{\mathrm{eq}}}{2}$.
Moreover if $k=n$ then $r=\frac{\mathcal{Q}_{\mathrm{eq}}}{2}$.

Corollary

$$N(\lambda)\sim\lambda^r\ln^k\lambda$$

(by the Karamata tauberian theorem)

Remark: We have $r\in\mathbb{N}^*$ under the condition (in privileged coordinates)

$$\forall q\in M\quad \forall\sigma\in\mathbb{R}^n\quad \forall i\in\{1,\dots,n\}\quad \forall k\in\{1,\dots,m\}\qquad w_i^\sigma(X_k)=w_i^\sigma(\widehat{X}_k^q)$$

(which is valid for instance when each X_k can be at any point q "factorized" by $\widehat{X}_k^q$)

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Some examples of singular sR structures

name	definition	asymptotics	concentration
k-Grushin	$X_1 = \partial_1, \quad X_2 = x_1^k \partial_2 \qquad (k \geqslant 1)$	$\frac{\vert\ln t\vert}{t}$ if $k=1$ $\frac{1}{t^{k+1}}$ if $k \geqslant 2$	$N = \mathcal{S} = \{x_1 = 0\}$
Sing. k-Grushin	$X_1 = \partial_1, \quad X_2 = (x_1^k - x_2)\partial_2$ $(k \geqslant 2)$	$\frac{\vert\ln t\vert}{t} \quad \forall k \geqslant 2$	$N = \mathcal{S} = \{x_2 = x_1^k\}$
	$X_1 = \partial_1, \quad X_2 = (x_1^{2p} + x_1 y_1^k)\partial_2$ $p, k \in \mathbb{N}^*$	$\frac{\ln^2 t}{t}$ if $k=1$ $\frac{1}{t^{p+\frac{1}{2}-\frac{2p-1}{2k}}}$ if $k \geqslant 2$	$N = \{(0,0)\}$ $\subset \mathcal{S} = \{x_1^{2p} + x_1 y_1^k = 0\}$
	$X_1 = \partial_1, \quad X_2 = (x_1^2 - x_2^3)\partial_2$	$\frac{1}{t^{7/6}}$	$N = \{(0,0)\} \subsetneq \mathcal{S} = \{x_1^2 = x_2^3\}$
Martinet	$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1^2 \partial_3$	$\frac{\vert\ln t\vert}{t^2}$	$N = \mathcal{S} = \{x_1 = 0\}$
Nilp. tang. hyp.	$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1^2 x_2 \partial_3$	$\frac{\ln^2 t}{t^2}$	$N = \{x_1 = x_2 = 0\}$ $\subsetneq \mathcal{S} = \{x_1 x_2 = 0\}$
Ghezzi Jean	$X_1 = \partial_1$ $X_2 = \partial_2 + x_1\partial_3 + x_1^2\partial_5$ $X_3 = \partial_4 + (x_1^k + x_2^k)\partial_5 \quad (k \geqslant 2)$	$\frac{1}{t^{7/2}}$ if $k=2$ $\frac{\vert\ln t\vert}{t^{7/2}}$ if $k=3$ $\frac{1}{t^{2+\frac{k}{2}}}$ if $k \geqslant 4$	$N = \mathbb{R}^5 \supsetneq \mathcal{S} = \{x_1 = x_2 = 0\}$ $N = \mathcal{S} = \{x_1 = x_2 = 0\}$ $N = \mathcal{S} = \{x_1 = x_2 = 0\}$

It may happen that $N = \mathcal{S}$, $N \subsetneq \mathcal{S}$ or $N \supsetneq \mathcal{S}$.

Quantum Limits (QL)

- M compact, smooth volume μ
- $\triangle$ selfadjoint nonnegative operator on $L^2(M,\mu)$, with compact resolvent
  - discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty$
  - $(\phi_j)_{j\in\mathbb{N}^*}$ orthonormal eigenbasis of $L^2(M,\mu)$

A quantum limit (on the base) is a (weak) limit of the sequence of probability measures $|\phi_j^2|\, d\mu$.

More generally (pseudo-diff. version), a QL is a probability measure on $S^\star M$, closure point of the sequence of measures $\mu_j(a) = \langle \mathrm{Op}(a)\phi_j, \phi_j\rangle$ (a: symbol of order 0)

General question in quantum physics, quantum chaos: what are the possible QLs?

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Quantum Ergodicity (QE)

We say we have QE for $(T, (\phi_n)_{n\in\mathbb{N}^*})$ if there exist a probability measure ν on M and a subsequence $(n_j)_{j\in\mathbb{N}^*}$ of density one such that

$$|\phi_{n_j}|^2\, d\mu \rightharpoonup d\nu \qquad \text{as } j \to +\infty$$

(density one meaning that $\frac{1}{n}\#\{j \mid n_j \leqslant n\} \underset{n\to+\infty}{\longrightarrow} 1$)

More generally (pseudo-diff. version): $\left\langle \mathrm{Op}(a)\phi_{n_j}, \phi_{n_j} \right\rangle_{L^2(M,\mu)} \to \int_\Sigma a\, d\tilde{\nu} \qquad \forall a \in \mathcal{S}^0$

Shnirelman Theorem (1974)

On (M, g) compact Riemannian manifold, if the geodesic flow is ergodic, then we have QE for any orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $\triangle$, with $\nu =$ normalized Riemannian volume (and $\tilde{\nu} =$ Liouville measure on S^*M).

(Zelditch 1987, Colin de Verdière 1985, Gérard Leichtnam 1993, Zelditch Zworski 1996)

The 3D contact case

(M, D, g) 3D contact sR structure, M compact

$D = \ker \alpha_g$ with $(d\alpha_g)_{|D}$ = oriented volume form induced by g on D.
Reeb vector field Z: $\alpha_g(Z) = 1$ and $d\alpha_g(Z, \cdot) = 0$. Popp $= |\alpha_g \wedge d\alpha_g|$.

Theorem (Colin de Verdière, Hillairet, Trélat, Duke Math. 2018)

If the Reeb flow is ergodic on M for the Popp measure, then we have QE.

We identify $S^\star M = U^\star M \cup S\Sigma$, with $U^\star M = \{g^\star = 1\}$ (cylinder bundle).

Without any ergodicity assumption:

1. $\forall \beta \in QL \qquad \beta = \beta_0 + \beta_\infty \qquad$ with $\quad \beta_0 \perp \beta_\infty \qquad$ and
   - $\mathrm{supp}(\beta_0) \subset U^\star M$, and β_0 invariant under the sR geodesic flow
   - $\mathrm{supp}(\beta_\infty) \subset S\Sigma$, and β_∞ invariant under the (lift to $S\Sigma$ of the) Reeb flow
2. $\exists (n_j)_{j\in\mathbf{N}^*}$ of density one s.t. $\forall \beta \in QL$ associated with $(\phi_{n_j})_{j\in\mathbf{N}^*}$, we have $\mathrm{supp}(\beta) \subset S\Sigma$ (i.e., $\beta_0 = 0$)

A general path towards QE

(see Zelditch)

$N(\lambda) = \#\{n \mid \lambda_n \leqslant \lambda\}$

First step: establish a *microlocal Weyl law*

(and identify the invariant measure ν)

$$E(A) \overset{def}{=} \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} \langle A\phi_n, \phi_n\rangle = \bar{a} = \int_{S^\star M} a\, dW_\triangle$$

$\forall A \in \Psi^0$ with $a = \sigma_P(A)$.

($E(A)$ = Cesáro mean)

$\to$ Cesáro convergence property, under weak assumptions (without ergodicity):

$$\langle (A - \bar{a}\,\mathrm{id})\phi_n, \phi_n\rangle \to 0 \quad \text{in Cesáro mean}$$

A general path towards QE

(see Zelditch)

$N(\lambda) = \#\{n \mid \lambda_n \leqslant \lambda\}$

Second step: prove a *variance estimate*

$$V(A - \bar{a}\,\mathrm{id}) \overset{def}{=} \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} |\langle (A - \bar{a}\,\mathrm{id})\phi_n, \phi_n\rangle|^2 = 0$$

i.e.

$$|\langle (A - \bar{a}\,\mathrm{id})\phi_n, \phi_n\rangle|^2 \to 0 \quad \text{in Cesáro mean}$$

$\to$ Combine the microlocal Weyl law with ergodicity properties of some associated classical dynamics and with an Egorov theorem.

A general path towards QE

(see Zelditch)

End of the proof of QE:

Lemma (Koopman and Von Neumann)

Given a bounded sequence $(u_n)_{n\in\mathbb{N}}$ of nonnegative real numbers:

$$\frac{1}{n}\sum_{k=0}^{n-1} u_k \underset{n\to+\infty}{\longrightarrow} 0 \quad \Longleftrightarrow \quad \exists (n_j)_{j\in\mathbb{N}^*} \text{ of density one s.t. } \quad u_k \underset{j\to+\infty}{\longrightarrow} 0$$

Hence, there exists a density-one sequence $(n_j)_{j\in\mathbb{N}^*}$ s.t.

$$\lim_{j\to+\infty} \left\langle A\phi_{n_j}, \phi_{n_j} \right\rangle = \bar{a}.$$

Conclusion with a diagonal argument, using the fact that $\mathcal{S}^0$ admits a countable dense subset.

Classical and quantum Birkhoff normal forms

Model: Compact 3D flat Heisenberg group

Locally: $X_H = \partial_x$ and $Y_H = \partial_y - x\partial_z$, $\quad g$ flat

$Z_H = [X_H, Y_H]$ (Reeb)

$\triangle_H = X_H^2 + Y_H^2$

$g_H^* = \sigma(-\triangle_H) = h_{X_H}^2 + h_{Y_H}^2 = \rho_H I_H$

with $\rho_H = |h_{Z_H}|$ and $I_H = \left(\frac{h_{X_H}}{\sqrt{|h_{Z_H}|}}\right)^2 + \left(\frac{h_{Y_H}}{\sqrt{|h_{Z_H}|}}\right)^2$

By quantization ($R_H = \mathrm{Op}(\rho_H)$ and $\Omega_H = \mathrm{Op}(I_H)$):

$$\boxed{-\triangle_H = R_H \Omega_H = \Omega_H R_H}$$

General 3D contact case:

Melrose classical normal form:

$$\boxed{g_H^* \circ \chi = g^*} \ (= \sigma_P(-\triangle))$$

with χ symplectic diffeo
(valid globally along any Reeb orbit)

Quantum normal form near Σ:

$$\boxed{-\triangle = R\Omega + V_0 + \mathrm{O}_\Sigma(\infty)}$$

with $[R, \Omega] = 0$, V_0 of order 0

Perspectives, open questions

- 3D contact case: are the Reeb periods spectral invariants?
- 5D contact case: resonances $\rightarrow$ Birkhoff normal form only at finite order along Σ (Cyril Letrouit, ongoing)
- QE (and QLs) in more general cases:
  - Grushin: we have QE if the singular curve is connected.
  - Martinet: ergodicity of the singular flow (in the Martinet surface) $\Rightarrow$ QE?
  - Quasi-contact in dim 4: magnetic lines = projections of singular geodesics. Ergodicity of the magnetic vector field $\Rightarrow$ QE? (Nikhil Savale, ongoing)
- Microlocal Weyl law $W_\triangle$ in general singular cases?
- Controllability, observability of subelliptic wave equations (Cyril Letrouit, ongoing)
- Trace formulas in sR geometry (Nikhil Savale, ongoing)