The Hermite calculus of isotropic states and applications

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Plan of the talk

Part I: Isotropic quantum states

Definition and examples
Their symbols
The symbol calculus
Complex polarizations

Part II: Hermite operators and applications

Hermite operators
Mixed states and a Szegő limit theorem
A refinement of Weyl's law ($X = T^*M$)

Settings and notation

(X, ω) a symplectic manifold

Quantization:

- Cotangent case: $X = T^*M$, $\mathcal{H} = L^2(M)$
  Quantum observables: $\hbar$-pseudodifferential operators.
- Kähler case (or almost): (X, ω, J)
  $\mathcal{L} \to X$ pre-quantum line bundle (assumed to exist),

$$\mathcal{H}_k \subset C^\infty(X, \mathcal{L}^k)$$

  space of holomorphic or near-holomorphic sections.
  Quantum observalbes: B-T operators.

Part I: Isotropic quantum states

Consider $\Sigma \subset X$ *isotropic*:

$$(T\Sigma)^{\circ} \supset T\Sigma.$$

An associated class of $\hbar$-dependent wavefunctions?

Microlocal setting (distributions): $X = T^*M \setminus \{0\}$, Σ conic.

- Hermite distributions of Boutet de Monvel and Guillemin:
  - V. Guillemin: *Symplectic spinors and PDEs*, 1975.
  - L. Boutet de Monvel and V. Guillemin: *The spectral theory of Toeplitz operators*, 1981.
- Closely related: R. Melrose: *Marked Lagrangian distributions*, unpublished preprint.

Semi-classical setting:

- Simplest example: $\Sigma = \{\mathbf{x}_0\}$, a single point. $H : X \to \mathbb{R}$ having $\mathbf{x}_0$ as non-degenerate global minimum. The *ground states* of $Op(\mathcal{H})$ concentrate at $\mathbf{x}_0$ and are examples.
- Quasi-modes associated to a periodic trajectory: (Ralston 1976, Colin de Verdière 1977, many others)
- A general class was defined by Paul and U. (1995)
- Systematic theory: Guillemin, U. and Wang (2016 and in progress)

Simplest examples: General coherent states

When the coisotropic is a point, $\Sigma = \{\mathbf{x}_0\}$, associated functions are called *general coherent states* centered at $\mathbf{x}_0$.

In case $X = T^*\mathbb{R}^n$: If $\mathbf{x}_0 = (x_0, p_0)$,

$$\psi_\hbar(x) = e^{i\hbar^{-1}(x - p_0)}\, a\left(\frac{x - x_0}{\sqrt{\hbar}}\right) \in L^2(\mathbb{R}^n), \quad a \in \mathcal{S}$$

is a general coherent state.

Complex polarizations: In case $X = \mathbb{C}^n$ and if $\mathbf{x}_0$ is the origin,

$$\psi(z) = e^{z^T A z/2\hbar}\, e^{-|z|^2/2\hbar} \in \text{Bargmann space}$$

with A complex symmetric with e-vals $|\lambda| < 1$.

“Squeezed” coherent states (in Bargmann):

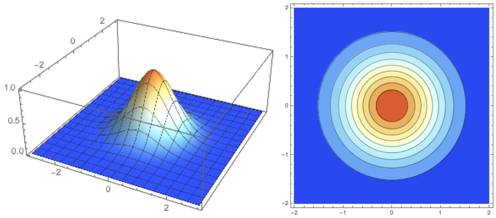


Figure: $|\psi|^2$, $n = 1$, $A = 0$: Standard Coherent State

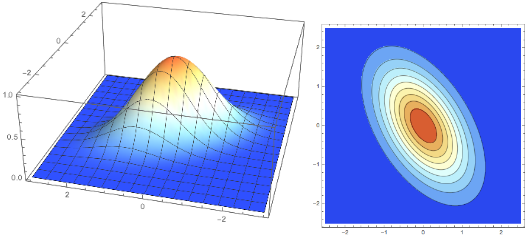


Figure: $|\psi|^2$, $n = 1$, $A = -\frac{1}{4} + \frac{i}{2}$, $|\alpha| = 0.8$

Model case for $X = T^*M$, $\mathcal{H} = L^2(M)$:

Split $M = \mathbb{R}^k \times \mathbb{R}^l$, $x = (t, u)$, and let

$$\Sigma = \{u = 0\} \cap \text{ zero section.}$$

Model isotropic wave functions:

$$\psi_\hbar(t, u) \sim \sum_{j=0}^{\infty} \hbar^{j/2}\, a_j\left(t, \frac{u}{\sqrt{\hbar}}\right), \quad a_j(t, \cdot) \in \mathcal{S}(\mathbb{R}^l)$$

Gaussian beams or “coherent states in u with parameters t”

Definition: The symbol of $\psi_\hbar$ is $a_0(t, \cdot)\sqrt{dt}$.

If (τ, μ) are dual variables to (t, u), (u, μ) are coordinates on the *symplectic normal* of Σ_0

$$\mathcal{N}^\Sigma := \Sigma^\circ / \Sigma$$

In more detail:

$(t, u, \tau, \mu) \in T^*\mathbb{R}^n$,

$$\Sigma = \operatorname{span}\{\partial_t\}, \quad \Sigma^\circ = \operatorname{span}\left\{\partial_t,\, \partial_u,\, \partial_\mu\right\},$$

$$\mathcal{N} = \Sigma^\circ/\Sigma \cong \operatorname{span}\left\{\partial_u,\, \partial_\mu\right\}.$$

Recall: Symbol is $\sqrt{dt}\, a_0(t, \cdot)$

For each t,

$$u \mapsto a_0(t, u)$$

is a Schwartz function in a quantization of $\mathcal{N}$

Equivalent general definitions (real polarizations):

- Circle bundle approach (Paul-U): Relates the distributional and semi-classical approaches.
- Apply $\hbar$-FIOs associated to canonical transformations to the model case.
- Integral representations ($X = T^*M$):

$$\Psi_\hbar(x) = \int_{\mathcal{T}\times U} e^{i\hbar^{-1} f(x,t,u)} a(x, t, \hbar^{-1/2} u;\ \hbar)\, dtdu$$

$$\Sigma = \{(x, d_x f)\ ;\ d_t f = 0,\ d_u f = 0, u = 0\}$$

- The class can be characterized by stable regularity under certain ideal of ΨDOs

$$\boxed{\text{Get spaces } \mathcal{I}^m(\Sigma),\ m = \text{order in } \hbar}$$

An important example: Szegő kernels

$$(L, \nabla) \to (X, \omega), \qquad \mathcal{H}_k \subset L^2(X, L^k)$$

$$\Pi_k : L^2(X, L^k) \to \mathcal{H}_k, \quad \Pi_k(\cdot, \cdot) \in C^\infty(L^k \boxtimes (L^*)^*)$$

- Case X Kähler : $\mathcal{H}_k$ = holomorphic sections of L^k
  1. Boutet de Monvel - Sjostrand '76
  2. Boutet de Monvel - Guillemin '81;
  3. Zelditch '98
- Almost Kähler case: $\mathcal{H}_k$ = span of low-lying eigensections of $\Box_k$
  1. Guilemin - U Asympt. Analysis '88
  2. Ma - Marinescu Adv. Math. '08,
  3. Bortwhick - U TAMS 2007: We proved that the projectors Π_k have the same semi-classical structure as in the Kähler case.

Symbolic calculus of isotropic wave functions

- The symbol at a point $\sigma \in \Sigma$ is of the form

$$\sqrt{d\sigma} \otimes \varphi, \quad \varphi \in \mathbb{H}^{\infty}_{\mathcal{N}_\sigma}$$

  φ is a "Schwartz function", the Schwartz factor and $\sqrt{d\sigma}$ a half form on Σ.

- Complication: The Schwartz factor can be hard to keep track of.

- $\mathbb{H}^{\infty}_{\mathcal{N}_\sigma}$ is the space of smooth vectors of the Heisenberg representation of the symplectic normal (–details later–)

- The calculus is with respect to the action of ΨDOs and FIOs.

The abstract Hilbert space of an Mp vector space

A metaplectic vector space E (think fiber of $\mathcal{N}^{\Sigma}$) has an associated abstract Hilbert space

$$E \rightsquigarrow \mathbb{H}_E$$

where the Schrödinger $\hbar = 1$ representation of E is realized.

How?

- Real polarizations $L \subset E$, complex positive J, $\rightsquigarrow$ Hilbert space $\mathcal{H}_L$, $\mathcal{B}_J$ (Bargmann space).

- $\mathcal{H}_L$, $\mathcal{B}_J$: Sections of a pre-quantum line bundle covariant constant w.r.t. the polarization, tensored by a transverse half form.

- If $g : E \to E$ is an Mp map, by translations get unitary isomorphism
$$U_g : \mathcal{H}_L \to \mathcal{H}_{g(L)}.$$

- **The Blattner-Kostant-Sternberg pairing:** If L, $L' \subset E$ are lagrangians, the BKS pairing, in this case, is a unitary
$$V_{L',L} : \mathcal{H}_L \to \mathcal{H}_{L'}$$

- One can then define the abstract Hilbert space associated with E as:

$$\mathbb{H}_E = \{(\psi, L)\ ;\ L \subset E \text{ lagrangian and } \psi \in \mathcal{H}_L\} / \sim$$

  where

$$(\psi, L) \sim (\psi', L') \quad \Leftrightarrow \quad \psi' = V_{L',L}(\psi).$$

- If $g : E \to E$ an Mp map, can form

$$\rho(g) : \mathcal{H}_L \xrightarrow{U_g} \mathcal{H}_{g(L)} \xrightarrow{V_{g(L),L}} \mathcal{H}_L.$$

  $g \mapsto \rho(g)$ is the Mp representation

- $\mathbb{H}_E$ carries the rep. of the Heisenberg group of (E, ω).

  $\mathbb{H}_E^\infty$ is the space of smooth vectors (Schwartz functions).

Back to an isotropic $\Sigma \subset X$:

- Apply the above to the fibers of the symplectic normal bundle $\mathcal{N}^\Sigma \to \Sigma$.

- One can talk about an infinite-rank "bundle",

$$\mathcal{S} \to \Sigma$$

whose fiber at $s \in \Sigma$ is $\mathbb{H}^\infty_{\mathcal{N}^\sigma_s}$

$$\mathcal{S}_s = \mathbb{H}^\infty_{\mathcal{N}^\sigma_s}$$

- The symbol of $\psi \in \mathcal{I}(\Sigma)$ is a section of the bundle

$$\mathcal{S} \otimes \wedge^{1/2} T\Sigma \to \Sigma.$$

- Simplest example: Coherent state centered at $\Sigma = (0,0) \in \mathbb{R}^{2n}$,

$$\psi = a\left(x/\sqrt{\hbar}\right), \quad \sigma(\psi) = a.$$

Symbol calculus under ΨDOs:

- If $\psi \in I^m(\Sigma)$ and $\mathcal{A}$ is a quantum observable (order zero),

$$\mathcal{A}(\psi) \in I^m(\Sigma) \quad \text{and} \quad \sigma(\mathcal{A}(\psi)) = A\,\sigma(\psi).$$

- If $A|_\Sigma = 0$ then $\mathcal{A}(\psi) \in I^{m-1/2}(\Sigma)$. ξ_A defines a section of $\mathcal{N}^\Sigma$ and its symbol is the Heisenberg action of that section on $\sigma(\psi)$.

- If $A \in \mathcal{I}(\Sigma)$ then $\mathcal{A}(\psi) \in I^{m-1}(\Sigma)$, and its symbol is a (kind of) Lie derivative of $\sigma(\psi)$ with respect to ξ_A.

Calculus under the action of FIOs

1. Under an FIO associated to a canonical transformation:
   Translate the Schwartz factor by the transformation.
   Must keep track of polarizations!
2. Under an FIO associated to a canonical relation:
   Theorem:
   Clean intersection condition $\Rightarrow$ image is in the class
   Symbol: Complicated recipe in general.
   Will only consider special cases.

Complex polarizations

Sections associated with Σ:

$$(X, \omega, J) \text{ (almost) Kähler, } \mathcal{L} \to X$$

- $\mathcal{L}^* \supset Z \to X$ unit circle bundle,

$$\mathcal{B}_k = H^0(X, \mathcal{L}^k) \quad \text{or its A. K. analogue,}$$

- $\Pi_k : L^2(X, \mathcal{L}^k) \to \mathcal{B}_k \subset C^\infty(Z)$
- $\Sigma \subset X$ B - S, $Z \supset \widetilde{\Sigma}$ horiz. lift
- Consider:

$$\psi_k := \Pi_k(u), \qquad u \in I^{\cdot}(Z, \mathcal{C}\Sigma),$$

$$\mathcal{C}\Sigma := \{(p, r\alpha_p) \; ; \; r > 0, \; p \in \widetilde{\Sigma}\}.$$

Ref: Ioos 2018, BPU 1999 (Lagrangian case)

These are special cases of the real-polarization case

Example:

$$X = \mathbb{R}^{2n} \cong \mathbb{C}, \qquad z = \frac{1}{\sqrt{2}}(p + iq),$$

$$\mathcal{B}_k = \left\{ \psi(z) = f(z)\, e^{-k|z|^2/2} \;;\; \bar{\partial} f = 0 \text{ and } \psi \in L^2 \right\}.$$

$\Pi_k : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_k$ is "twisted convolution" with standard Gaussian

$$\Pi_k(u, v) = \left(\frac{k}{\pi}\right)^n e^{ik\omega(v,u)/2}\, e^{-k|u-v|^2/2}.$$

Projecting $e^{kv^T \mathcal{M} v/2} \in L^2(\mathbb{R}^{2n})$, $\mathcal{M}$ $2n \times 2n$ symmetric, $\Re \mathcal{M} < 0$ yields

$$C\, e^{z^T Az/2\hbar}\, e^{-|z|^2/2\hbar}$$

A of size $n \times n$, $A^* A < I$.

Natural space for the symbols (complex polarization)

$$(X, \omega, J), \qquad \Sigma \subset X \text{ isotropic}$$

$$\mathcal{N} = \Sigma^\circ/\Sigma \cong \Sigma^\circ \cap J(\Sigma^\circ) \subset TX$$

which is the maximal complex subspace of Σ°.

The Schwartz part of the symbol of a state associated with Σ naturally lives on the Bargmann space

$$\mathcal{B}(\Sigma^\circ \cap J(\Sigma^\circ))$$

New examples: Squeezed spin states

Joint work with J. Rousseva

- Natural isotropic states associated with points of $X = \mathbb{CP}^n$ (squeezed c.s.)
- Regard $\mathbb{CP}^n$ as the reduction of $\mathbb{C}^{n+1}$ at $|z| = 1$
- Quantum reduction: Project

$$\mathcal{R}_k : \mathcal{B}_k(\mathbb{C}^{n+1}) \longrightarrow \underbrace{\mathcal{B}_{k,k}(\mathbb{C}^{n+1})}_{\text{homog. polyn. degree } k=1/\hbar} \cong \mathcal{B}_k(\mathbb{CP}^n)$$

- Project a general Gaussian c.s. with center $w \in S^{2n-1}$:

$$\varphi_{w,A} = \mathcal{R}_k\left(\psi_{A,w}\right),$$

$$\psi_{A,w}(z) = e^{kQ_A(z-w)/2}\, e^{kz\overline{w}}\, e^{-k(|z|^2+|w|^2)/2},$$

$$Q_A(v) = vAv^T, \qquad AA^* < I$$

The symbol of $\varphi_{w,A}$ is the reduction of the symbol of $\psi_{A,w}$

- The symbol of $\psi_{A,w}$ is in $\mathcal{B}(T_w\mathbb{R}^{2n+2})$,
- Reduction with respect to $V := T_w S^{2n-1} \subset \mathbb{R}^{2n+2}$ yields a push-forward map

$$\mathcal{B}^\infty(T_w\mathbb{R}^{2n+2}) \to \mathcal{B}^\infty(T_{\pi(w)}\mathbb{CP}^n)$$

(dual of “quantization commutes with reduction”)

Concrete formulas: $n = 1$

- The squeezed $SU(2)$ states centered at the origin of the Bloch sphere:

$$C_k \sum_{0 \le \ell \le k/2} \left(\frac{1}{2k}\right)^{\ell} \frac{1}{\sqrt{(k-2\ell)!}} \sqrt{\binom{2\ell}{\ell}} \mu^{\ell} \, |k - 2\ell\rangle$$

$$|\mu| < 1.$$

- This is the reduction of

$$\psi(z_1, z_2) = e^{k\mu \, z_2^2} \, e^{-k(|z|^2+1)/2}.$$

- Have norm estimates. Propagation is built into the theory.

Part II
Hermite operators and applications

Hermite operators

Let

$$\Lambda \subset X \quad \text{submanifold of dimension } d,$$

$$\Lambda \overset{\Delta}{\times} \Lambda := \{(\lambda, \lambda) \; ; \; \lambda \in \Lambda\}.$$

Since

$$\Lambda \overset{\Delta}{\times} \Lambda \subset \text{ diagonal} \subset X \times X^{-},$$

$\Lambda \overset{\Delta}{\times} \Lambda$ is an isotropic submanifold of $X \times X^{-}$.

Definition: An Hermite operator associated with Λ is an operator whose Schwartz kernel is an isotropic function associated with $\Lambda \overset{\Delta}{\times} \Lambda$.

$$\mathcal{O}^{m}(\Lambda)$$

Note that

$$\left(\Lambda \overset{\Delta}{\times} \Lambda\right) \circ \left(\Lambda \overset{\Delta}{\times} \Lambda\right) = \Lambda \overset{\Delta}{\times} \Lambda.$$

Theorem:

$$\mathcal{O}^{m}(\Lambda) \circ \mathcal{O}^{m'}(\Lambda) \subset \mathcal{O}^{m+m'-n+d-1/2}(\Lambda)$$

Corollary: The symbols of operators in $\mathcal{O}^{m}(\Lambda)$ form an algebra.

The algebraic structure of the symbol algebra depends on the "symplectic nature" of Λ

Example: The case of a point

Let

$$\Lambda = \left\{(0,0) \in \mathbb{R}^{2n}\right\}.$$

Tthe operator ρ_a on $L^2(\mathbb{R}^b)$ with kernel

$$\mathcal{K}_{\rho_a}(x,y) = a\left(\frac{x}{\sqrt{\hbar}}, \frac{y}{\sqrt{\hbar}}\right), \quad a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$$

is an Hermite operator associated with Λ.
The symbol of $\rho_a \circ \rho_b$ is

$$\sigma_{\rho_a \circ \rho_b}(x,y) = \int a(x,u)b(u,y)du$$

The symbol calclus is non-commutative, $\sigma_{\rho_a \circ \rho_b} \neq \sigma_{\rho_b \circ \rho_a}$.

A co-isotropic example

Split the variables $\mathbb{R}^n \ni \xi = (\xi', \xi'')$;

$$\Lambda = \{(x; \xi', \xi'' = 0)\} \subset T^*\mathbb{R}^n.$$

$$\mathcal{K}_{\rho_a}(x, y) = \int e^{i\hbar^{-1}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi', \frac{\xi''}{\sqrt{\hbar}}\right) d\xi,$$

where a is Schwartz with respect to ξ''-variables.

$$\sigma_{\rho_a} = a(x, \xi', \cdot), \qquad \text{and}$$

$$\sigma_{\rho_a \circ \rho_b} = \sigma_{\rho_a} \sigma_{\rho_b},$$

$$a\#b - b\#a = \frac{\sqrt{\hbar}}{i}\{a, b\}'' + \frac{\hbar}{i}\{a, b\}' + O(\hbar^{3/2}).$$

About the symbol of an Hermite operator:

(E,ω) a symplectic vector space, $\Lambda \subset E$ a subspace.

$$\Lambda_\Delta = \Lambda \overset{\Delta}{\times} \Lambda = \{(\lambda,\lambda)\ ;\ \lambda \in \Lambda\},$$

$$\mathcal{N} := (\Lambda_\Delta)^\circ / \Lambda_\Delta$$

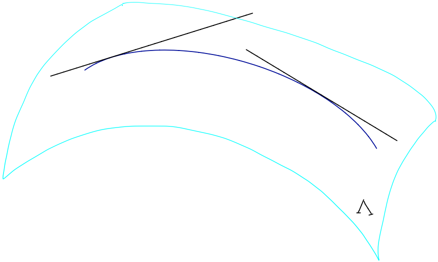
the symplectic normal

Recall: **The Schwartz factors of the symbol live in $\mathbb{H}^\infty_{\mathcal{N}}$**

Then $\exists$ isomorphism

$$\mathcal{N} \cong \Lambda^\circ \times (\Lambda^\circ)^* \cong T^*\Lambda^\circ.$$

In case Λ is co-isotropic



$$\mathcal{N} \cong T^*\Lambda^\circ = \text{cotangent bundle of the null fibers,}$$

$$\mathbb{H}^\infty = \mathcal{S}\Big((\Lambda^0)^*\Big) = \text{Schwartz functions on cotangent spaces}$$

<u>Theorem:</u> In this polarization, the symbol calculus for the composition of Hermite operators is just multiplication.

The symbol calculus is commutative

Application 1: Calculus of certain mixed states

X Kähler , $\Lambda \subset X$.
$d\lambda$ normalized Riemannian density,

$$\int_\Lambda d\lambda = 1.$$

Let

$$\rho = \int_\Lambda |z\rangle\langle z|\, d\lambda(z), \quad |z\rangle = \Pi_k(\cdot, z)$$

where $|z\rangle$ is a *standard* coherent state at z is a *mixed state*:

$$\rho \geq 0 \quad \text{and} \quad \mathrm{tr}(\rho) = 1.$$

In case Λ is Lagrangian, these were considered by Y. LeFloch (2017).

More generally, can insert an amplitude $a : \Lambda \to \mathbb{R}^+$

$$\rho_a = \int_\Lambda |z\rangle\langle z| \, a(z) d\lambda(z).$$

Up to a factor of $\dim \mathcal{B}_k$, the Berezin transform is included:

$$\Lambda = \text{diagonal} \subset \underbrace{X \times X^-}_{\text{new } X}, \quad a = \Pi_k(z, z)$$

Theorem

ρ_a is an Hermite operator associated with Λ, The symbol is the ground state of the symplectic normal.

This allows us to use the symbol calculus to prove e.g.

- (LeFloch) If $\mathcal{A}$ is a quantum observable, then

$$\mathrm{tr}(\mathcal{A}\rho) \sim \int_{\Lambda} A\, a\, d\sigma \quad \text{as} \quad \hbar \to 0.$$

- The time t evolution of ρ,

$$e^{it\hbar^{-1}\mathcal{A}} \rho\, e^{-it\hbar^{-1}\mathcal{A}},$$

  is a mixed state of the same type as ρ but associated to

$$\Lambda_t = \phi_t(\Lambda), \quad \phi_t : X \to X \text{ flow of } A.$$

A Szegő limit theorem

After joint work w. S. Pérez-Esteva

$$\Lambda \subset X \text{ co-isotropic}, \quad d\sigma = \text{Riemannian volume}$$

$$a : \Lambda \to \mathbb{R}_+.$$

Let $R > 0$ such that $[0, R]$ contains the spectrum of ρ_a for all k, and the image of a.

Let φ be a function such that

$$\exists p > 0 \qquad \frac{\varphi(t)}{t^p} \quad \text{is continuous on } \; [0, R].$$

Then

For a suitable normalization

$$S_a = Ck^{d/2-n}\rho_a,$$

$$\lim_{k\to\infty}\left(\frac{2\pi}{k}\right)^{d/2} \operatorname{tr}(\varphi(S_a)) = \int_\Lambda \mathcal{O}_{-d'/2}(\varphi)(a(w))\, d\sigma(w), \quad d' = 2n - d$$

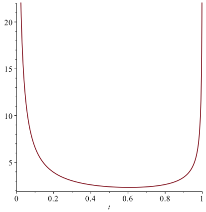
where $\mathcal{O}_{-\alpha}$

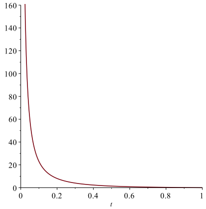
$$\mathcal{O}_{-\alpha}(\varphi)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \varphi(s) \log(t/s)^{\alpha-1} \frac{ds}{s}$$

Key:

$$\operatorname{tr}(S_a^\ell) \sim C_n k^{d/2} \frac{1}{\ell^{d'/2}} \int_\Lambda a^\ell \, d\sigma$$

Density $d' = 1$, $d' = 4$ $(a \equiv 1)$





Entropy

For a mixed state ρ (co-isotropic case)

$$\mathbb{H}(\rho_a) := -\sum_{j=1}^{\infty} p_j \, \log(p_j), \quad \{p_j\} = \text{spec}\, \rho_a$$

Theorem:

$$\lim_{k \to \infty} \left[\mathbb{H}(\rho_a) + \log(C_d k^{-d/2}) \right] =$$

$$-\frac{1}{\Gamma(d'/2)} \int_\Gamma \left(\int_0^{a(w)} s \log(s) \, \log\left(\frac{a(w)}{s}\right)^{\frac{d'}{2}-1} \, \frac{ds}{s} \right) d\sigma(w).$$

Application 2: A refinement of Weyl's law

$P = \frac{1}{2}\hbar^2\Delta + V$ on M compact Riemannian,

$$H : T^*M \to \mathbb{R}, \quad H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x)$$

Theorem Assume 0 is a regular value of H and $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Then

$$P_\varphi := \varphi\left(\frac{1}{\sqrt{\hbar}} P\right)$$

is an Hermite operator associated with

$$\Lambda := H^{-1}(0).$$

The Schwartz factor of symbol is $\varphi \in \mathcal{S}\left[(T_\lambda\Lambda^\circ)^*\right]$, where

$$(T_\lambda\Lambda^\circ)^* \cong \mathbb{R} \quad \text{using the basis dual to} \quad \Xi_H$$

Taking traces:

Let

$$P(\psi_j) = E_j \psi_j, \quad \{\psi_j\} \text{ an ONB,}$$

Then

$$\mathrm{tr} P_\varphi = \sum_j \varphi\left(\frac{E_j}{\sqrt{\hbar}}\right) \sim \frac{\sqrt{\hbar}}{(2\pi\hbar)^n} |\Lambda| \int_{\mathbb{R}} \varphi(s)\, ds,$$

$$|\Lambda| = \text{Liouville measure of } \Lambda,$$

which leads to

$$\boxed{\# \left\{ j\, ;\, |E_j| \leq c\, \sqrt{\hbar} \right\} \sim \frac{2c\sqrt{\hbar}}{(2\pi\hbar)^n} |\Lambda|}$$

A more speculative application: Quantization of co-isotropics

- Hermite operator calculus is commutative.
- Model case:

$$\Lambda = \{(x; \xi', \xi'' = 0)\} \subset T^*\mathbb{R}^n,$$

null leaves: $(x', \xi') =$ const, parametrized by x''

- A symbol is *basic* iff it's independent of x''
- In general

$$a\#b - b\#a = \frac{\sqrt{\hbar}}{i}\{a, b\}'' + \frac{\hbar}{i}\{a, b\}' + O(\hbar^{3/2}).$$

- If a, b are basic

$$a\#b - b\#a = \frac{\hbar}{i}\{a, b\}' + O(\hbar^{3/2})$$

So *basic symbols get quantized*.

Thank you