Probabilistic Analogues in Kähler analysis Quantization in Symplectic Geometry Universität Köln, 2019

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Probability measures arising in Kähler geometry

This talk is a survey of results on sequences $\{\mu_k\}_{k=1}^{\infty}$ of probability measures arising in Kähler geometry. The parameter k corresponds to the power L^k of a positive Hermitian line bundle $L \to M$. There are two types:

- Toric case: $\{\mu_k^x\}$ are prob measures on $\mathbb{Z}^m \cap kP$, the lattice points in the kth dilate of a Delzant polytope P, where $x \in P$. These are generalization of multi-nomial distributions and satisfy many of the same properties.

- General case: $\{\mu_k^z\}_{k=1}^{\infty}$ are prob measures on $\mathbb{R}$, pointwise spectral measures for a Toeplitz operator. The toric case is the one where there are $m = \dim M$ commuting operators.

Classical results in probability

The sequences μ_k should be compared with the sequence of convolution powers μ^{*k} of a probability measure μ on $\mathbb{R}^m$. The convolution $\mu * \nu$ of two probability measures is defined by

$$\mu * \nu(E) = \int_{\mathbb{R}^n} \mu(E - x)\nu(dx). \tag{1}$$

Convolution powers arise when one studies sums $\sum_{j=1}^k X_j$ of i.i.d. random variables with values in $\mathbb{R}^m$. Three (or four) classical results involve limits of dilates of μ^{*k}. By a dilate we mean $D_t\mu(E) = \mu(tE)$.

- The weak LLN (law of large numbers): $D_{k*}\mu^{*k} \rightharpoonup \delta_m$, where $m = \int x d\mu$ is the mean;
- The CLT (central limit theorem): If μ is re-centered to have mean zero, and normalized to have variance 1, then $D_{\sqrt{k}*}\mu^{*k} \rightharpoonup N(0,1)$.
- The Cramer LDP (large deviations principle: measures exponential decay of $D_k\mu_k^*\{x : |x - m| \geq C\}$.
- McMillan entropy theorem (later).

CLT

Suppose μ is a probability measure on $\mathbb{R}^n$ such that $\int d\mu = 1, \int |x|^2 d\mu < \infty$ and $\int x d\mu = 0$. Consider $\mu_k = k^{n/2} D_{\sqrt{k}} \mu^{*k}$ where $D_{\sqrt{k}}(x) = x\sqrt{k}$. Suppose that

$$\int x_i x_j d\mu = A_{ij}.$$

Then $\mu_k \to \gamma_A$ where $\gamma_A = \frac{1}{\sqrt{|\det A|}} e^{-\langle A^{-1}x, x\rangle} dx$.

LDP

Let $E \subset \mathbb{R}$, let $\overline{E}$ denotes its closure and let E^o denote its interior. $\{\mu_n\}$ satisfies an LDP if :

$$\begin{aligned} (UB) \quad & \limsup_{n\to\infty} \frac{1}{n} \log\left(\frac{S_n}{n} \in E\right) \leq -\inf_{\overline{E}} I(x), \\ (LB) \quad & \liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{S_n}{n} \in E\right) \geq -\inf_{E^o} I(x). \end{aligned} \tag{2}$$

I.e.

$$\begin{aligned} (UB) \quad & \limsup_{n\to\infty} \frac{1}{n} \log\left(\mu_n(nE)\right) \leq -\inf_{\overline{E}} I(x), \\ (LB) \quad & \liminf_{n\to\infty} \frac{1}{n} \log\left(\mu_n(nE)\right) \geq -\inf_{E^o} I(x). \end{aligned} \tag{3}$$

Cramer LDP

Let μ be a probability measure on $\mathbb{R}$ and let $\mu_k = \mu^{*k}$. Let $M_\mu(\xi) = \int_{\mathbb{R}^n} e^{x\cdot\xi} d\mu(x)$ be the moment generating function and let $\Lambda_\mu(\xi) = \log M_\mu(\xi)$ be the logarithmic moment generating function. Let $I_\mu(x) = \sup_{\xi\in\mathbb{R}}(x\xi - \Lambda_\mu(\xi)$ be its Legendre transform. Cramer LDP:

$$D_N \mu^{*N}[a,\infty] \leq e^{-NI_\mu(a)}, \quad a \in [m,\infty],$$

$$D_N \mu^{*N}[-\infty,a] \leq e^{-NI_\mu(a)}, \quad a \in [-\infty,m].$$

Bernoulli and Binomial distributions

The simplest example of the classical CLT is that of the Bernoulli measures $\mu_p = (1-p)\delta_0 + p\delta_1$ and their convolution powers on the unit interval $[0,1]$. The kth convolution power $\mu_p^{*k} = 2^{-k} \sum_{n=0}^{k} p^k (1-p)^{n-k} \binom{k}{n} \delta_n$ has its support in $[0,k]$.

Convolution of binomial distributions

$B(n, p)$ is the measure

$$\mu_{n,p} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k.$$

Consider $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Then

$$\mu * \mu = \tfrac{1}{4}(\delta_0 + 2\delta_1 + \delta_2),$$

$$\mu * \mu * \mu = \tfrac{1}{8}(\delta_0 + 3\delta_1 + 3\delta_2 + \delta_3),$$

$$\mu^{*n} = \tfrac{1}{2^n}(\delta_0 + \tbinom{n}{1}\delta_1 + \tbinom{n}{2}\delta_2 + \cdots + \tbinom{n}{n}\delta_n).$$

One then dilates back to $[0, 1]$ to get

$$D_n \mu^* n = \frac{1}{2^n}(\delta_0 + \binom{n}{1}\delta_{\frac{1}{n}} + \binom{n}{2}\delta_{\frac{2}{n}} + \cdots + \binom{n}{n}\delta_1).$$

The measure peaks when $k = \binom{n}{n/2}$ at the point $\frac{1}{2}$.

LLN, CLT and LDP for binomial measures

In the law of large numbers one rescales the measure back to $[0,1]$ as $2^{-k}\sum_{n=0}^{k} p^k(1-p)^{n-k}\binom{k}{n}\delta_{\frac{n}{k}}$, which tends weakly to δ_p. In the CLT one recenters the measure at 0 and then dilates it by $\sqrt{k}$ so that it spreads out to $[-\sqrt{k},\sqrt{k}]$, and then it tends to the Gaussian of mean 0 and variance 1. The parameter $p \in [0,1]$ of μ_p is analogous to the parameter $z \in M$ in the Kähler setting. In the special case of $\mathbb{CP}^1$ with the Fubini-Study metric, the measures are precisely the Bernoulli measures μ_p with $p \in [0,1]$ being the image of z under the moment map. Moreover, the CLT is in fact the classical CLT in this special case, i.e. μ_k^z is the kth convolution power of μ_1^z.
The LDP is the Cramer LDP: $P(S_n \geq an) \leq e^{-nI_p(a)}$, where $I_p(a) = a\log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p}$ is the relative entropy of the a and the p binomomial measures.

Bergman kernels, partial Bergman kernels, spectral projections kernels

The probability measures in the Kähler setting do not arise (in general) as convolution powers. They are constructed from Bergman kernels and spectral projections. We now introduce our notation: Let $(L, h) \to (M, \omega)$ be a positive Hermitian line bundle over a Kähler manifold. The kth Bergman kernel is the orthogonal projection:

$$\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k) := \text{ holomorphic sections of } \mathrm{L}^{\mathrm{k}}.$$

Its kernel w.r.t the Kähler volume form is denoted $\Pi_{h^k}(x, y)$. For any such kernel, the metric contraction (density of states) is denoted (in terms of an ONB),

$$\Pi_{h^k}(x) := \sum_{j=1}^{N_k} |s_{k,j}(z)|^2_{h^k}, \quad N_k = \dim H^0(M, L^k)$$

Toeplitz Hamiltonians

Let $H \in C^\infty(M, \mathbb{R})$. Quantize H as the self-adjoint zeroth order Toeplitz operator

$$H_k := \Pi_k(\frac{i}{k}\nabla_{\xi_H} + H)\Pi_k : H^0(M, L^k) \to H^0(M, L^k) \qquad (4)$$

acting on the space $H^0(M, L^k)$ of holomorphic sections. Here, ξ_H is the Hamiltonian vector field of H, ∇_{ξ_H} is the Chern covariant deriative on sections, and H acts by multiplication.
Denote the eigenvalues by $\mathrm{Sp}(\hat{\mathrm{H}}_\mathrm{k}) := \{\mu_{\mathrm{k,j}}\}$ and the eigenspaces by

$$V_{\mu_{k,j}} := \{s \in H^0(M, L^k) : H_k s = \mu_{k,j} s\}. \qquad (5)$$

We denote by $\Pi_{k,j} : H^0(M, L^k) \to V_{\mu_{k,j}}$ the orthogonal projection to $V_{\mu_{k,j}}$. Denote its metric contraction (DOS) by $\Pi_{k,j}(z)$.

Sequences in Kähler analysis

For any $(L, h) \to (M, \omega)$ and $H : M \to \mathbb{R}$ we define three sequences analogous to $\mu^{*k}, D_{\sqrt{k}}\mu^{*k}, D_k\mu^{*k}$:

$$\begin{cases} (i) & d\mu_k^z(x) \;\; = \sum_j \Pi_{k,j}(z)\delta_{\mu_{k,j}}(x), \\ \\ (ii) & d\mu_k^{z,\frac{1}{2}}(x) = \sum_j \Pi_{k,j}(z)\delta_{\sqrt{k}(\mu_{k,j}-H(z))}(x), \\ \\ (iii) & d\mu_k^{z,1,\tau}(x) = \sum_j \Pi_{k,j}(z)\delta_{k(\mu_{k,j}-H(z))+\sqrt{k}\tau}(x), \end{cases} \tag{6}$$

We view these scalings as analogous to three scalings of the convolution powers μ^{*k} of a probability measure μ supported on $[-1, 1]$ (say). The third scaling (iii) corresponds to μ^{*k}, which is supported on $[-k, k]$. The first scaling (i) corresponds to the Law of Large Numbers, which rescales μ^{*k} back to $[-1, 1]$. The second scaling (ii) corresponds to the CLT (central limit theorem) which rescales the measure to $[-\sqrt{k}, \sqrt{k}]$.

Allowed and forbidden regions for a spectral interval

Let E be a regular value of H. We denote the partial Bergman kernels for the corresponding spectral subspaces by

$$\mathcal{S}_k := \mathcal{H}_{k,E} := \bigoplus_{\mu_{k,j} < E} V_{\mu_{k,j}}, \tag{7}$$

The allowed, resp. forbidden region for these subspaces are,

$$\mathcal{A} := \{z : H(z) < E\}, \quad \mathcal{F} = \{z : H(z) > E\}, \quad \mathcal{C} = \{z : H(z) = E\}. \tag{8}$$

The partial Bergman kernel is,

$$\Pi_{k,E} : H^0(M, L^k) \to \mathcal{H}_{k,E}, \tag{9}$$

Weak LLN and Pointwise Weyl

THEOREM

Fix a regular value E of $H : M \to \mathbb{R}$. Then,

$$d\mu_k^z \rightharpoonup \delta_{H(z)}.$$

I.e. for any $f \in C^\infty(\mathbb{R})$, we have

$$\Pi_k(z)^{-1} \int_{-\infty}^{E} f(\lambda) d\mu_k^z(\lambda) \to \begin{cases} f(H(z)) & \text{ifz} \in \mathcal{A} \\ 0 & \text{ifz} \in \mathcal{F}. \end{cases} \tag{10}$$

In particular, the density of states of the partial Bergman kernel is given by the asymptotic formula:

$$\Pi_k(z)^{-1}\Pi_{k,E}(z) \sim \begin{cases} 1 \mod O(k^{-\infty}) & \text{ifz} \in \mathcal{A} \\ 0 \mod O(k^{-\infty}) & \text{ifz} \in \mathcal{F}. \end{cases} \tag{11}$$

where the asymptotics are uniform on compact sets of $\mathcal{A}$ or $\mathcal{F}$.

Sequences in Toric Kähler analysis

In the toric case, instead of one Toeplitz Hamiltonian, we have n commuting Toeplitz Hamiltonians, the generators of a Hamiltonian torus action $\mathbf{T}^m$, whose joint eigenvalues are the lattice points $\alpha \in kP$ in the kth dilate of a polytope.
By definition, $\exists$ a Hamiltonian torus action $\Phi^{\vec{t}}(z) : \mathbf{T}^m \times M \to M$ which extends holomorphically to a $(\mathbb{C}^*)^m$ action, and M is the closure of an open orbit $M^o = (\mathbb{C}^*)^m\{z_0\}$. Let h denote a $\mathbf{T}^m$-invariant Hermitian metric on L with curvature form ω. The moment map

$$\mu_h := \mu : M \to P \subset \mathbb{R}^m, \tag{12}$$

defines a torus bundle on the open orbit over a convex lattice (Delzant) polytope P. There is a natural basis $\{s_\alpha\}_{\alpha \in kP}$ of the space $H^0(M, L^k)$ of holomorphic sections of the k-th power of L by eigensections s_α of the $\mathbf{T}^m$ action. In a standard frame e_L of L over M^o, they correspond to monomials z^α on $(\mathbb{C}^*)^m$.

The probability measures

For any $z \in M^o$ and $k \in \mathbb{N}$, we define the probability measure,

$$\mu_k^z = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} \delta_{\frac{\alpha}{k}} \in \mathcal{M}_1(\mathbb{R}^m), \tag{13}$$

on $\mathbb{R}^m$. Here, $\|s_\alpha\|_{h^k}$ is the L^2 norm of s_α with respect to the natural inner product $\mathrm{Hilb}_k(h)$ induced by the Hermitian metric on $H^0(M, L^k)$ and $\Pi_{h^k}(z,z)$ is the contracted Szegö kernel on the diagonal (or density of states). The measures are discrete measures supported on $P \cap \frac{1}{k}\mathbb{Z}^m$.

Note: μ_k^z depends only on $\mu_h(z) \in P$. These are generalizations of multi-nomial measures. They are vector-valued analogues of μ_k^x for one $\hat{H}_k$ (which are prob measures on $\mathbb{R}$ rather than on $P \subset \mathbb{R}^m$).

Main results (joint with Peng Zhou)

- For one Toeplitz Hamiltonian $\hat{H}_k$, the measures $d\mu_k^z(x) \ = \sum_j \Pi_{k,j}(z)\delta_{\mu_{k,j}}(x)$ satisfy a weak LLN, a CLT, and (in the real analytic case) an LDP.
- In the toric case, the measures μ_k^z satisfy a weak LLN, a CLT, an LDP.
- (new result with Pierre Flurin) In the toric case, the entropy of the discrete measures μ_k^z have asymptotic expansions. For a discrete measure, the (Shannon) entropy is $H(\mu) := -\sum_\alpha \mu(\alpha) \log \mu(\alpha)$

Possibly there also exist entropy asymptotics in the single Toeplitz Hamiltonian case.

Why do these analogue results exist

- In very rare cases such as $(M, \omega) = (\mathbb{CP}^m, \omega_{FS})$, the sequences of toric measures μ_k^z on the simplex really are convolution powers. In fact, they are multinomial distributions. In the non-compact Bargmann-Fock space, they are Poisson distributions. Probably this requires all powers h^k of the metric to be *balanced*.

- The measures μ_k^z are the laws (distribution measures) of random variables $X_k^{z,\mathbb{R}}$, resp. $X_k^{z,\mathbb{R}^m}$, which take values $\mu_{k,j}$, resp $\alpha \in kP \cap \mathbb{Z}^m$, with probabilities $\frac{\Pi_{k,j}(z)}{\Pi_k(z)}$, resp. $\frac{\Pi_{k,\alpha}}{\Pi_k(z)}$. They seem to behave as sums of k i.i.d. variables. But we have no definition of these.

Results in the toric case

To determine the appropriate Gaussian measure we need to determine the asymptotics as $k \to \infty$ of the mean,

$$\vec{m}_k(z) = \int_P \vec{x} d\mu_k^z(x),$$

resp. the covariance matrix

$$[\Sigma_k]_{ij}(z) = \int_P (x_i - m_{k,i}(z))(x_j - m_{k,j}(z)) d\mu_k^z.$$

Lemma

Let $\mu_h : M \to P$ be the moment map (12). Then,

$$\vec{m}_k(z) = \mu_h(z) + O(1/k), \quad \Sigma_k(z) = \frac{1}{k} Hess\, \varphi(z) + O(\frac{1}{k^2})$$

Normalizing the measures to have mean zero and variance one

We re-center the measures at $\mu(z)$, i.e. put

$$\tilde{\mu}_k^z = \mu_k^z(x - \mu_h(z)),$$

and then dilate by $\sqrt{k}$ to obtain the normalized sequence,

$$D_{\sqrt{k}}\tilde{\mu}_k^z = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \, \delta_{\sqrt{k}(\frac{\alpha}{k} - \mu_h(z))}. \tag{14}$$

Equivalently, if $f \in C_b(\mathbb{R}^m)$. Then,

$$\langle f, D_{\sqrt{k}}\tilde{\mu}_k^z \rangle = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \, f(\sqrt{k}(\frac{\alpha}{k} - \mu_h(z)), \tag{15}$$

Here, $C_b(\mathbb{R}^m)$ denotes the space of bounded continuous functions on $\mathbb{R}^m$.

Weak LLN for toric measures

PROPOSITION

Let $\mu_0 : M \to P$ be the moment map with respect to the symplectic form ω_0. Then for any $z \in M$,

$$\mu_k^z \rightharpoonup \delta_{\mu_0(z)}.$$

Thus,

$$\mu_0(z) = \lim_{k\to\infty} \frac{1}{\Pi_{h^{kd}}(z,z)} \sum_{\alpha \in kP} (\frac{\alpha}{k}) \ \frac{||s_\alpha(z)||^2_{h_0^k}}{\mathcal{Q}_{h_0^k}(\alpha)}.$$

CLT for toric Kähler manifolds

Theorem

In the topology of weak convergence on $C_b(\mathbb{R}^m)$,*

$$D_{\sqrt{k}}\widetilde{\mu}_k^z \stackrel{w*}{\to} \gamma_{0,Hess\ \varphi(z)}.$$

That is, for any $f \in C_b(\mathbb{R}^m)$,

$$\int_{\mathbb{R}^m} f(x) D_{\sqrt{k}} d\widetilde{\mu}_k^z(x) \to \int_{\mathbb{R}^m} f(x) d\gamma_{0,Hess\ \varphi(z)}(x).$$

The role of the parameter z is similar to that of the parameter p in the Bernoulli measures $\mu_p = p\delta_0 + (1-p)\delta_1$ and their convolution powers on the unit interval $[0,1]$. In very special cases, such as the Fubini-Study metric h of $M = \mathbb{CP}^m$, μ_k^z is itself a sequence of dilated convolution powers, $\mu_k^z = (\mu_1^z)^{*k} = \mu_1^z * \mu_1^z \cdots * \mu_1^z$ (k times).

Density of states for a toric sub-polytope PBK

The following graphics are from B. Shiffman-S.Z. on partial Bergman kernels in the toric setting. The CLT is the Gaussian transition.

Entropy of the toric measures μ_k^z

The main result is an asymptotic formula for the entropy $H(\mu_k^z)$ as $k \to \infty$. There are very few results, even classical, on asymptotic entropy.

For a finite probability distribution $\{p_\alpha\}$, the entropy of the distribution is

$$H = -\sum_\alpha p_\alpha \ln p_\alpha.$$

Thus, the entropy of μ_k^z is

$$H(\mu_k^z) = -\sum_{\alpha \in kP} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \ln \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}}.$$

Entropy $H(\mu)$ of a discrete probability measure μ is a measure of the degree to which μ is uniform. The larger the entropy, the more uniform the measure. Thus, entropy of μ_k^z is a measure of its uniformity as a measure on $kP \cap \mathbb{Z}^m$.

Asymptotics of entropy (joint with Pierre Flurin)

Theorem

Let a_0 (a universal constant) be the leading term of the density of states $\lim_{m\to\infty} k^{-m}\Pi_m(z)$ As $k\to\infty$. Then, in dimension m,

$$\begin{aligned} H(\mu_k^z) &= \tfrac{m}{2}\log(k) + \tfrac{3m}{2}\log(2\pi) - \log(a_0) + \tfrac{1}{2}\log(|\det \nabla^2 u_0(x_0)|) \\ &\quad + \tfrac{(2\pi)^{-m} m}{2a_0}|\det \nabla^2 u_0(x_0)|^{-1} \end{aligned}$$

Note that the leading order term is of order $\log k$ and is rather trivial. The geometry is in the constant order term. In the case of sums of i.i.d. real-valued random variables, i.e. convolution powers of probability measures on $\mathbb{R}$, Dyachkov proved that

$$H(\mu^{*k}) \simeq \frac{1}{2}(\log k) + \frac{1}{2}\log(2\pi e\sigma^2) + o(1).$$

Large deviations principle and entropy asymptotics

The entropy asymptotics are based on an LDP due to J.Song and S.Z.(2010) : that μ_k^z satisfy a large deviations principle (LDP). Heuristically, an LDP means that the measure $\mu_k^z(A)$ of a Borel set A is obtained asymptotically by integrating $e^{-kI^z(x)}$ over A, where I^z is known as the rate functional and k is the rate. The rate functions I^z for $\{d\mu_k^z\}$ depend on whether z lies in the open orbit M^o of M or on the divisor at infinity $\mathcal{D}$; equivalently, they depend on whether the image $\mu_0(z)$ of z under the moment map for ω_0 lies in the interior P^o of the polytope P or along a face F of its boundary ∂P.

Precise definition

A function $I : E \to [0, \infty]$ is called a rate function if it is proper and lower semicontinuous. A sequence μ_k $(k = 1, 2, \dots)$ of sequence of probability measures on a space E is said to satisfy the *large deviation principle with the rate function I* (and with the speed k) if the following conditions are satisfied:

(1) The level set $I^{-1}[0, c]$ is compact for every $c \in \mathbb{R}$.

(2) For each closed set F in E,
$\limsup_{k\to\infty} \frac{1}{k} \log \mu_k(F) \leq -\inf_{x\in F} I(x)$.

(3) For each open set U in E,
$\liminf_{k\to\infty} \frac{1}{k} \log \mu_k(U) \geq -\inf_{x\in U} I(x)$.

Heuristically, in the sense of logarithmic asymptotics, the measure μ_k is a kind of integral of $e^{-kI(x)}$ over the set.

Laplace LDP

Dupuis-Ellis gave an alternative definition in terms of Laplace type integrals. . Put:

$$F(z,h) = -\inf_{x\in P}(h(x) + I^z(x)). \tag{16}$$

Then $d\mu_k^z$ satisfies the *Laplace principle on P with rate function I^z uniformly on M* if, for all compact subsets $K \subset M$ and all $h \in C_b(P)$ we have:

(1) For all $c \in \mathbb{R}$, $\bigcup_{z\in M}(I^z)^{-1}[0,c]$ is compact for every $c \in \mathbb{R}$.

(2) For each $h \in C_b(P)$,
$\limsup_{k\to\infty}\sup_{z\in M}\left(\frac{1}{k}\log\int_P e^{-kh}d\mu_k^z - F(z,h)\right) \leq 0.$

(3) For each $h \in C_b(P)$,
$\liminf_{k\to\infty}\inf_{z\in M}\left(\frac{1}{k}\log\int_P e^{-kh(x)}d\mu_k^z(x) - F(z,h)\right) \geq 0.$

The upper and lower bounds of course imply, for each $h \in C_b(P)$,

$$\lim_{k\to\infty}\sup_{z\in M}\left(\frac{1}{k}\log\int_P e^{-kh(x)}d\mu_k^z(x) - F(z,h)\right) = 0.$$

The LDP

Theorem

For any $z \in M$, the probability measures μ_k^z satisfy a uniform Laplace large deviations principle with rate k and with convex rate functions $I^z \geq 0$ on P defined as follows:

- *If $z \in M^0$, the open orbit, then $I^z(x) = u_0(x) - \langle x, \log|z| \rangle + \varphi(z)$, where φ is the canonical Kähler potential of the open orbit and u_0 is its Legendre transform, the symplectic potential;*
- *When $z \in \mu_0^{-1}(F)$ for some face F of ∂P, then $I^z(x)$ restricted to $x \in F$ is given by $I^z(x) = u_F(x) - \langle x', \log|z'| \rangle + \varphi_F(z)$, where $\log|z'|$ are orbit coordinates along F, φ_F is the canonical Kähler potential for the subtoric variety defined by F and u_F is its Legendre transform. On the complement of $\bar{F}$ it is defined to be $+\infty$.*
- *When z is a fixed point then $I^z(v) = 0$ and elsewhere $I^z(x) = \infty$.*

Spectral subspaces

We now return to scalar Hamiltonians and their Toeplitz quantizatins $\hat{H}_k$ on general Kähler manifolds. The sequences of probability measures are defined by partial Bergman kernels, namely the projections

$$\Pi_{h^k,[E_1,E_2]} : H^0(M, L^k) \to \mathcal{H}_{k:[E_1,E_2]}. \tag{17}$$

onto the subspaces

$$\mathcal{S}_k := \mathcal{H}_{k:[E_1,E_2]} := \bigoplus_{\mu_{k,j} \in H^{-1}([E_1,E_2])} V_{\mu_{k,j}} \tag{18}$$

where $\mu_{k,j}$ are the eigenvalues of $\hat{H}_k$ and

$$V_k(\mu_{k,j}) := \{s \in H^0(M, L^k) : \hat{H}_k s = \mu_{k,j} s\}. \tag{19}$$

Single Toeplitz operators on Kähler manifolds

For any $(L, h) \to (M, \omega)$ and $H : M \to \mathbb{R}$ we define three sequences of probability measures on $\mathbb{R}$ analogous to $\mu^{*k}, D_{\sqrt{k}}\mu^{*k}, D_k\mu^{*k}$:

$$\begin{cases} (i) & d\mu_k^z(x) = \sum_j \Pi_{k,j}(z)\delta_{\mu_{k,j}}(x), \\ (ii) & d\mu_k^{z,\frac{1}{2}}(x) = \sum_j \Pi_{k,j}(z)\delta_{\sqrt{k}(\mu_{k,j}-H(z))}(x), \\ (iii) & d\mu_k^{z,1,\tau}(x) = \sum_j \Pi_{k,j}(z)\delta_{k(\mu_{k,j}-H(z))+\sqrt{k}\tau}(x), \end{cases} \tag{20}$$

The weak LLN says that $d\mu_k^z \to \delta_{H(z)}$. The variance is of order $\frac{1}{k}$. Next we give the CLT.

Interface result for smoothed partial Bergman kernel in a $\frac{1}{\sqrt{k}}$ tube around a regular level

The CLT pertains to the family of measures $d\mu_k^{z,\frac{1}{2}}(x) = \sum_j \Pi_{k,j}(z)\delta_{\sqrt{k}(\mu_{k,j}-E)}(x)$ when z lies in $k^{-\frac{1}{2}}$-tube around $\{H = E\}$, i.e. $|H(z) - E| = O(k^{-\frac{1}{2}})$. We let Φ^β denote the *gradient flow* of H, moving us off $\{H = E\}$ in the normal direction. Then $z = \Phi^{\beta/\sqrt{k}} z_0$.

Since it is a weak* convergence result, we let $f \in C_b(\mathbb{R})$, let $z = \Phi^{\beta/\sqrt{k}} z_0$ and consider

$$\langle f, d\mu_k^{z,\frac{1}{2}} \rangle = \sum_j f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(\Phi^{\beta/\sqrt{k}} z_0).$$

ERF

$\mathrm{Erf(x)} = \int_{-\infty}^{\mathrm{x}} \mathrm{e}^{-\mathrm{s}^2/2} \frac{\mathrm{ds}}{\sqrt{2\pi}}$ is the cumulative distribution function of the Gaussian, i.e., $\mathbb{P}_{X\sim N(0,1)}(X < x)$. The usual Gaussian error function $\mathrm{erf(x)} = (2\pi)^{-1/2} \int_{-\mathrm{x}}^{\mathrm{x}} \mathrm{e}^{-\mathrm{s}^2/2} \mathrm{ds}$ is related to Erf by $\mathrm{Erf(x)} = \frac{1}{2}(1 + \mathrm{erf}(\frac{\mathrm{x}}{\sqrt{2}}))$.

THEOREM

$$\left(\frac{\Pi_{k,\mathcal{S}_k}}{\Pi_k}\right)(\Phi(z, t/\sqrt{k})) = Erf(2\sqrt{\pi}t) + O(k^{-1/2}), \qquad (21)$$

The CLT

THEOREM

If $z_0 \in \{H = E\}$, a non-critical level, then there exists a complete asymptotic expansion,

$$\sum_j f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(\Phi^{\beta/\sqrt{k}} z_0) \simeq k^m I_m(f, E) + k^{m-\frac{1}{2}} I_{m-\frac{1}{2}}(f, E) + \cdots ,$$

in descending powers of $k^{\frac{1}{2}}$, with leading coefficient

$$[I_m(f, E) = \lim_{k \to \infty} k^{-m} \sum_{j : \mu_{k,j} \in P_0} f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(F^{\beta/\sqrt{k}} z_0)$$

$$= \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{2x\sqrt{\pi}}{|\nabla H(z_0)|} - \beta \frac{|\nabla H(z_0)|}{\sqrt{\pi}}\right)^2} \frac{2dx}{\sqrt{2|\nabla H|(z_0)}},$$

a Gaussian measure centered at $\beta|\nabla H(z_0)|$.

CLT

The usual CLT would consider $D_{\sqrt{k}} T_{-H(z)} \mu_k^z$, i.e.

$$\sum_j f(\sqrt{k}(\mu_{k,j} - H(z)))\Pi_{k,j}(z).$$

Effectively, this puts $z \in H^{-1}(E)$ and puts $\beta = 0$, so

$$\int f d D_{\sqrt{k}} T_{-H(z)} \mu_k^z = \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{2x\sqrt{\pi}}{|\nabla H(z)|}\right)^2} \frac{2dx}{\sqrt{2|\nabla H|(z}}.$$

The variance is $|\nabla H(z)|^2$.

Interface result for smoothed partial Bergman kernel in a $k^{-1/4}$ tube around a singular level

The formula in the smooth case is un-defined if $\nabla H(z_0) = 0$, i.e. if the point z_0 is a critical point of H. In this case one gets a degenerate Gaussian. Its type depends on what kind of critical point z_0 is. The next result gives the CLT when z_0 is a non-degenerate (Morse) critical point. E.g. a non-degenerate minimum point, the ground state of the Hamiltonian.

In this case, we need to use a different scaling in the normal direction. Integrated against a test function $f \in C_b(\mathbb{R})$ we consider

$$\Pi_{k,E,f,1/2}(z_c + k^{-1/4}u) := \sum_j \Pi_{k,j}(z_c + k^{-1/4}u) \cdot f(k^{1/2}(\mu_{k,j} - E))$$

rather than

$$\sum_j f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(\Phi^{\beta/\sqrt{k}} z_0).$$

Interface result for smoothed partial Bergman kernel in a $k^{-1/4}$ tube around a singular level

THEOREM

Let z_c be a non-degenerate Morse critical point of H, $E = H(z_c)$, $u \in T_{z_c}M$. Then for any $f \in \mathcal{S}(\mathbb{R})$, we have

$$\Pi_{k,E,f,1/2}(z_c + k^{-1/4}u) := \sum_j \Pi_{k,j}(z_c + k^{-1/4}u) \cdot f(k^{1/2}(\mu_{k,j} - E))$$

$$= k^m f(\mathrm{Hess}_{z_c} H(u,u)/2) + O_f(k^{m-1/4}).$$

In particular, the normalized rescaled pointwise spectral measure

$$d\hat{\mu}_k^{(z_c,u,1/4),1/2}(x) := \frac{\sum_j \|s_{k,j}(z_c + k^{-1/4}u)\|^2 \, \delta_{k^{1/2}(\mu_{k,j}-E)}(x)}{\sum_j \|s_{k,j}(z_c + k^{-1/4}u)\|^2}$$

converges weakly

$$\hat{\mu}_k^{(z_c,u,1/4),1/2}(x) \rightharpoonup \delta_{\frac{1}{2}\mathrm{Hess}_{z_c} H(u,u)}(x).$$

Brief idea of methods

Given a function $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space) one defines

$$f(\hat{H}_k) = \int_{\mathbb{R}} \hat{f}(\tau) e^{ik\tau \hat{H}_k} d\tau = \int_{\mathbb{R}} \hat{f}(t) U_k(t) dt, \tag{22}$$

to be the operator on $H^0(M, L^k)$ with the same eigensections as $\hat{H}_k$ and with eigenvalues $f(\mu_{k,j})$. Here,

$$U_k(t) = \exp itk\hat{H}_k. \tag{23}$$

is the unitary group on $H^0(M, L^k)$ generated by $k\hat{H}_k$. Thus, if $s_{k,j}$ is an eigensection of $\hat{H}_k$, then

$$f(\hat{H}_k)\hat{s}_{k,j} = f(\mu_{k,j})\,\hat{s}_{k,j} \tag{24}$$

Toeplitz quantization of maps

A key step in the analysis is to construct

$$U_k(t) = \exp ikt\hat{H}_k$$

ias a Toeplitz Fourier integral operator quantizing the Hamilton flow of H. .

PROPOSITION

(S.Z. $\simeq 1988$) $\hat{U}_k(t,x,y)$ is a semi-classical Fourier integral operator. There exists an analytic symbol $\sigma_{k,t}$ so that if $\pi(x) = z$, the unitary group (23) has the form

$$\begin{aligned} U_k(t,z,z) &= \hat{U}_k(t,x,x) := \hat{\Pi}_k(\hat{g}^{-t})^*\sigma_{k,t}\hat{\Pi}_k(x,x) \\ &= \hat{\Pi}_k e^{2\pi ik\int_0^t H(\exp sX_H(z))ds} \quad (\exp tX_H^h)^*\sigma_{k,t}\hat{\Pi}_k(x,x). \end{aligned} \tag{25}$$

Scaling in t and scaling in z

To prove the interface results we rescale $U_k(t, z, z)$ both in t and in z. For the CLT, we rescale to the kernel $U_k(\frac{t}{\sqrt{k}}, z, z)$. Since the relevant time interval The time interval is now 'infinitesimal' (of the order $k^{-1/2}$), and the result can be proved by linearizing. The smoothed interface asymptotics thus amount to the asymptotics of the dilated sums,

$$\sum_j f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(F^{\beta/\sqrt{k}}(z_0))$$

$$= \int_{\mathbb{R}} \hat{f}(t) e^{-iE\sqrt{k}t} U_k(t/\sqrt{k}, z_k, z_k) \frac{dt}{2\pi}$$

where $z \in \partial\mathcal{A} = H^{-1}(E)$ and where $\hat{f} \in L^1(\mathbb{R})$, so that the integral on the right side converges.

Time scaled propagator

We employ the Boutet-de-Monvel-Sjostrand parametrix to give an explicit formula for the right side.

Proposition

If $z_0 \in M$ such that $dH(z_0) \neq 0$, then for any $\tau \in \mathbb{R}$,

$$\hat{U}_k(\tau/\sqrt{k}, \hat{z}_0, \hat{z}_0) = \left(\frac{k}{2\pi}\right)^m e^{i\tau\sqrt{k}H(z_0)} e^{-\tau^2 \frac{\|\xi_H(z_0)\|^2}{4}} (1 + O(|\tau|^3 k^{-1/2})),$$

where the constant in the error term is uniform as τ varies over compact subset of $\mathbb{R}$.

End of proof

n the exponent, using $E = H(z)$, we get

$$\begin{aligned}
-iE\sqrt{k}t + it\sqrt{k}H(z_k) &= it\sqrt{k}(H(z_k) - H(z)) \\
&= it\sqrt{k}[g(\nabla H(z), \tfrac{\beta}{\sqrt{k}}\nabla H(z)) + O((\beta/\sqrt{k})^2) \\
&= it\beta\|\nabla H(z)\|^2 + O(|t|k^{-\frac{1}{2}})
\end{aligned}$$

Furtheremore, $-\frac{1}{4}|t\xi_H(z_k)|^2 = -\frac{1}{4}|t\xi_H(z)|^2 + O(k^{-\frac{1}{2}}|t|^2)$.
Hence

$$\begin{aligned}
I &= \left(\tfrac{k}{2\pi}\right)^m \int_{\mathbb{R}} \hat{f}(t) e^{it\beta\|\nabla H(z)\|^2 - \frac{1}{4}|t\nabla H(z)|^2} \tfrac{dt}{2\pi}[1 + O(k^{-\frac{1}{2}})] \\
&= \left(\tfrac{k}{2\pi}\right)^m \int_{t\in\mathbb{R}} \int_{x\in\mathbb{R}} f(x) e^{-ixt} e^{it\beta\|\nabla H(z)\|^2 - \frac{1}{4}|t\nabla H(z)|^2} \tfrac{dxdt}{2\pi}[1 + O(k^{-\frac{1}{2}} \\
&= \left(\tfrac{k}{2\pi}\right)^m \int_{x\in\mathbb{R}} f(x) e^{-\left(\frac{x}{\|\nabla H\|} - \beta\|\nabla H(z)\|\right)^2} \frac{dx}{\sqrt{\pi}\|\nabla H(z)\|}[1 + O(k^{-\frac{1}{2}})]
\end{aligned}$$

Boutet de Monvel-Sjostrand parametrix

Near the diagonal in $\partial D_h^* \times \partial D_h^*$, the Boutet de Monvel-Sjostrand parametrix is:

$$\hat{\Pi}(x,y) = \int_0^\infty e^{-\sigma\psi(x,y)}\chi(x,y)s(x,y,\sigma)d\sigma + \hat{R}(x,y). \qquad (26)$$

Here, $\chi(x,y)$ is a smooth cutoff to the diagonal; $s(x,y,\sigma)$ is a semi-classical symbol of order $m = \dim_{\mathbb{C}} M$. The phase ψ is constructed from the Kähler potential $\varphi(z)$ of ω_0 by

$$\psi(x,y) = \psi((z,\lambda),(w,\mu)) = 1 - \lambda\bar{\mu}e^{\varphi(z,\bar{w})} \qquad (27)$$

where $\varphi(z,\bar{w})$ is the analytic extension of $\varphi(z) = \varphi(z,\bar{z})$ into the complexification $M \times \bar{M}$ of M. Also,

$$s \sim \sum_{n=0}^{\infty} \sigma^{m-n} s_n(x,y) \qquad (28)$$

is an analytic symbol in the sense of Boutet de Monvel. Finally, the remainder term $\hat{R}(x,y)$ is real analytic in a neighborhood of the diagonal.

Osculating Bargmann Fock representations

At each $z \in M$ there is an osculating Bargmann-Fock or Heisenberg model associated to (T_zM, J_z, h_z). We denote the model Heisenberg Bergman kernel on the tangent space by

$$\Pi_{h_z,J_z}^{T_zM}(u,\theta_1,v,\theta_2) : L^2(T_zM) \to \mathcal{H}(T_zM, J_z, h_z) = \mathcal{H}_J. \qquad (29)$$

In K-coordinates with respect to a K-frame,

$$\Pi_{h_z,J_z}^{T_zM}(u,\theta_1,v,\theta_2) = \pi^{-m} e^{i(\theta_1-\theta_2)} e^{u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}$$

$$= \pi^{-m} e^{i(\theta_1-\theta_2)} e^{i\Im u\cdot\bar{v}-\frac{1}{2}(|u-v|^2}$$

Note that $\Im u \cdot \bar{v} = \omega(u,v)$.

Unknown probabilistic theorems

- An LDP for $\{\mu_k^z\} \subset \mathcal{M}_1(\mathbb{R})$. It would give the exponential decay rate for $\mu_k^z([E_1, E_2])$ when $z \notin [E_1, E_2]$. Why: We would need to understand

$$\frac{1}{k} \log \sum_{j : \mu_{k,j} \in [E_1, E_2]} \frac{\Pi_{k,j}(z)}{\Pi_k(z)}.$$

- The entropy asymptotics of $\{\mu_k^z\}$.

By comparison with the toric case, the problem is that we have no explicit formulae for $\Pi_{k,j}(z)$. The eigensections are not known. We need to understand much more than just norming constants. On $\mathbb{CP}^1$ one can use WKB or Bohr-Sommerfeld. The result reduces to the toric case when H is a perfect Morse function.