Uniform estimates and Toeplitz quantization on $\mathbb{C}^n$ or bounded symmetric domains

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Outline

- Lipschitz approximation and UC symbols
- 1. Application: Deformation estimates
- 2. Application: Toeplitz algebras over the two ball.

Lipschitz Approximation and UC symbols

On a metric space (X, d) we consider function spaces:

- $\mathrm{Lip}(X)$="Lipschitz continuous functions".
- $\mathrm{UC}(X)$="uniformly continuous functions".

Note

Both spaces may contain unbounded functions and

$$\mathrm{Lip}(X) \subset \mathrm{UC}(X). \qquad (*)$$

Define:

$$\mathrm{BUC}(X) = \text{"bounded functions in } \mathrm{UC}(X)\text{"}.$$

Question: *Is the inclusion $(*)$ uniformly dense? If so: how to uniformly approximate UC-functions by Lip-functions?*

A very general answer

Definition (Metrically Convex Space)

A metric space (X, d) is called metrically convex if:

Two closed balls $B(x, s)$ and $B(y, t)$ around $x \in X$ and $y \in X$ and with radii $s \geq 0$ and $t \geq 0$ intersect iff $d(x, y) \leq s + t$.

Example: Complete Riemannian manifolds are metrically convex.

Theorem

Let (X, d) be metrically convex. Then the space of all Lipschitz functions $\mathrm{Lip}(X)$ is uniformly dense in $\mathrm{UC}(X)$ [a]

$$\mathrm{UC}(X) = \mathrm{Lip}_c(X) = \textit{uniform closure of Lipschitz functions.}$$

[a] e.g. see: Y. Benyamini, J. Lindenstrauss, *Geometric non-linear functional analysis*, AMS Colloquium Publication vol. 48, 2000.

See **L.A. Coburn**, *Approximation by Lipschitz functions*, ArXiv 21.

Aim: *Approximation by real analytic Lipschitz functions. Explicit and with a control of the remainder in the case $X = \mathbb{C}^n$.*

With $t > 0$ consider the heat transform of a (suitable) $f : \mathbb{C}^n \to \mathbb{C}$:

$$\widetilde{f}^{(t)}(w) := \frac{1}{(4\pi t)^n} \int_{\mathbb{C}^n} f(w-z) e^{-\frac{\|z\|^2}{4t}} dv(z).$$

Semi-group-property: $\widetilde{\{\widetilde{f}^{(s)}\}}^{(t)} = \widetilde{f}^{(t+s)}$, (if defined).

Definition

Mean oscillation of f for $t > 0$ at $w \in \mathbb{C}^n$:

$$\begin{aligned} \mathrm{MO}_t(f,w) &:= \widetilde{|f|^2}^{(t)}(w) - |\widetilde{f}^{(t)}(w)|^2 \\ &= \widetilde{\left\{ |f - \widetilde{f}^{(t)}(w)|^2 \right\}}^{(t)}(w) \geq 0. \end{aligned}$$

Definition

The functions having bounded mean oscillation are given by:

$$\mathrm{BMO}_t^2(\mathbb{C}^n) := \left\{ f \;:\; \|f\|_{\mathrm{BMO}_t} := \sup_{z \in \mathbb{C}^n} \sqrt{\mathrm{MO}_t(f,z)} < \infty \right\}. \qquad (*)$$

Remarks:

- The spaces $(*)$ are linear and independent of $t > 0$. Hence we denote them by $\mathrm{BMO}^2(\mathbb{C}^n)$.
- $\|\cdot\|_{\mathrm{BMO}_t}$ depends on $t > 0$ and only defines a semi-norm.
- The following inclusions hold (for all $t > 0$)

$$\mathrm{BUC}(\mathbb{C}^n) \subset \mathrm{UC}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n, d\mu_t).$$

where $\quad d\mu_t(z) = (\pi t)^{-\frac{n}{2}} e^{-\frac{\|z\|^2}{t}} dv(z).$

In particular: $\mathrm{BMO}^2(\mathbb{C}^n)$ contains unbounded functions.

Bounded oscillation versus bounded mean oscillation

Definition: A continuous function $f \in C(\mathbb{C}^n)$ is of bounded oscillation, if there is $C > 0$ such that for all $z, w \in \mathbb{C}^n$:

$$|f(z) - f(w)| \leq C + C\|z - w\|.$$

$\mathrm{BO}(\mathbb{C}^n) =$ *functions of bounded oscillation.*

Lemma

One has the inclusion $\mathrm{BO}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n)$. *More precisely,*

$$\begin{array}{ccccc} \mathrm{BMO}^2(\mathbb{C}^n) & = & \mathrm{BO}(\mathbb{C}^n) & + & \mathrm{BA}(\mathbb{C}^n), \\ \uparrow & & \uparrow & & \uparrow \\ f & = & \widetilde{f}^{(t)} & + & (f - \widetilde{f}^{(t)}) \end{array}$$

where $\mathrm{BA}(\mathbb{C}^n) := \{f \in \mathrm{BMO}^2(\mathbb{C}^n) \;:\; \widetilde{|f|^2}^{(t)}$ *is bounded*$\}$.

More inclusions: For all $t > 0$:

$$\mathrm{BUC}(\mathbb{C}^n) \subset \mathrm{UC}(\mathbb{C}^n) \subset \mathrm{BO}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n, d\mu_t).$$

Bounded oscillation versus bounded mean oscillation

Lemma

Let $z, w \in \mathbb{C}^n$ *and* $f \in \mathrm{BMO}^2(\mathbb{C}^n)$, *then for all* $t > 0$:

$$\left|\widetilde{f}^{(t)}(z) - \widetilde{f}^{(t)}(w)\right| \leq 2\|f\|_{\mathrm{BMO}_t}\|z - w\|.$$

Conclusion: $f \in \mathrm{BMO}^2(\mathbb{C}^n) \quad \Longrightarrow \quad \widetilde{f}^{(t)} \in \mathrm{Lip}(\mathbb{C}^n)$.

Lemma

Let $t > 0$ *and* $f \in \mathrm{UC}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n)$, *then*

- $\widetilde{f}^{(t)} \in \mathrm{Lip}(\mathbb{C}^n)$,
- $f - \widetilde{f}^{(t)} \in \mathrm{BUC}(\mathbb{C}^n)$.

$\mathrm{Lip}(\mathbb{C}^n)$ *is the "difference" between* $\mathrm{UC}(\mathbb{C}^n)$ *and* $\mathrm{BUC}(\mathbb{C}^n)$:

$$\mathrm{UC}(\mathbb{C}^n) = \mathrm{Lip}(\mathbb{C}^n) + \mathrm{BUC}(\mathbb{C}^n)$$

In particular: If $f \in \mathrm{UC}(\mathbb{C}^n)$ is unbounded, then the heat transform $\widetilde{f}^{(t)}$ remains unbounded for all $t > 0$.

Theorem A (W.B., L.A. Coburn)

Let $f \in \mathrm{UC}(\mathbb{C}^n)$, then the heat transform $\{\tilde{f}^{(t)}\}_{t>0}$ defines a flow of real analytic functions in $\mathrm{Lip}(\mathbb{C}^n)$ with

$$\lim_{t \to 0} \tilde{f}^{(t)} = f$$

uniformly on $\mathbb{C}^n$. A Lipschitz constant of $\tilde{f}^{(t)}$ is:

$$C_t := t^{-\frac{1}{2}} \| f(\cdot\, 2\sqrt{t}) \|_{\mathrm{BMO}_{1/4}}.$$

In particular, the following inclusion is dense:

$$\mathrm{Lip}(\mathbb{C}^n) \cap \underset{\substack{\uparrow \\ \text{real analytic functions}}}{C^{\omega}(\mathbb{C}^n)} \subset \mathrm{UC}(\mathbb{C}^n).$$

Remark: In the theorem one can replace $\mathbb{C}^n$ by $\mathbb{R}^n$.

Heat transform and Berezin-Toeplitz quantization

Let $t > 0$ and again consider a family of Gaussian measures on $\mathbb{C}^n$.

$$d\mu_t(z) = \frac{1}{(\pi t)^{\frac{n}{2}}} e^{-\frac{\|z\|^2}{t}} dv(z).$$

Definition:

The Fock space is defined as:

$$H_t^2 := H^2(\mathbb{C}^n, \mu_t) := \mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu_t).$$

With the orthogonal projection

$$P^t : L^2(\mathbb{C}^n, \mu_t) \to H^2(\mathbb{C}^n, \mu_t)$$

and a symbol $f : \mathbb{C}^n \to \mathbb{C}$ the Toeplitz operator T_f^t is defined as:

$$T_f^t := P^t \underset{\substack{\uparrow \\ \text{multiplication by } f}}{M_f} : H_t^2 \to H_t^2.$$

Heat transform and Berezin Toeplitz Quantization

Toeplitz quantization and Berezin transform

Berezin-Toeplitz quantization: For all $t > 0$:

$$\{\text{functions on } \mathbb{C}^n\} \ni f \mapsto T_f^t = P^t M_f \in \{\text{operators on } H_t^2\}.$$

Berezin transform:

$$\{\text{operators on } H_t^2\} \ni A \mapsto \widetilde{A}^{(t)}(z) := \langle A k_z^t, k_z^t \rangle \in \{\text{functions on } \mathbb{C}^n\}.$$

Notation: We write:

$$k_z^t := \|K_t(\cdot, z)\|^{-1} K_t(\cdot, z), \qquad z \in \mathbb{C}^n$$

with K_t being the reproducing kernel of H_t^2.

Example: Let $A = T_f^t$, then $\widetilde{A}^{(t)} = \widetilde{f}^{(t)} =$ *heat transform on* $\mathbb{C}^n$.

Generalization to the ball (or BSD)

Remark:

If we replace $\mathbb{C}^n$ by the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ or any BSD $\Omega \subset \mathbb{C}^n$ equipped with the Bergman metric. Similar ideas apply.

Example

Consider the open unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. Let $\alpha > 0$ with $\lambda = n + 1 + \alpha > 0$. Then, with $g : \mathbb{B}^n \to \mathbb{C}$:

$$B_{n+1+\alpha}(g)(w) = \\ = \frac{1}{\pi^n} \frac{\Gamma(n+1+\alpha)}{\Gamma(\alpha+1)} \int_{\mathbb{B}^n} g(z) \frac{(1-\|w\|^2)^{n+1+\alpha}(1-\|z\|^2)^{\alpha}}{|1 - z \cdot \overline{w}|^{2(n+1+\alpha)}} dv(z).$$

In this case $B_{n+1+\alpha}(g)$ also is called α-Berezin transform of g.

Theorem (W.-B. and L.A. Coburn, 2012)

Let $\Omega \subset \mathbb{C}^n$ be a BSD of genus p equipped with the Bergman metric and let $f \in \mathrm{UC}(\Omega)$.

There is a family of integral transforms $\{B_\lambda(f)\}_{\lambda \geq p}$ defining a "flow" of real analytic functions in $\mathrm{Lip}(\Omega)$ with

$$\lim_{\lambda \to \infty} B_\lambda(f) = f$$

uniformly on Ω. The Lipschitz constant of $B_\lambda(f)$ is dominated by

$$C_\lambda := 2\sqrt{\frac{\lambda}{p}}\|f\|_{\mathrm{BMO}_\lambda}.$$

In particular, the inclusion $\mathrm{Lip}(\Omega) \cap C^\omega(\Omega) \subset \mathrm{UC}(\Omega)$ is dense.

Idea: Let $B_\lambda(f)$ be the Berezin transform of the Toeplitz operator T_f^λ acting on standard weighted Bergman spaces $\mathcal{A}_\lambda^2$.

1. Application: deformation estimates

Estimates in deformation quantization (M. Rieffel):

$$\lim_{t \to 0} \|T_f^t\|_t = \|f\|_\infty, \tag{1}$$

$$\lim_{t \to 0} \|T_f^t T_g^t - T_{fg}^t\|_t = 0, \tag{2}$$

$$\lim_{t \to 0} \|\frac{1}{it}[T_f^t, T_g^t] - T_{\{f,g\}}^t\|_t = 0, \tag{3}$$

where

- $t \sim \hbar > 0$ corresponds to Plancks constant $\cong$ weight,
- $\{f,g\} = \sum_{i=1}^n \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \bar{z}_i} - \frac{\partial f}{\partial \bar{z}_i}\frac{\partial g}{\partial z_i}$ = Poisson bracket.

Theorem (D. Borthwick, 1989)

(1) - (3) hold for $f, g \in C_b^{4n+6}(\mathbb{C}^n)$.

Proof: Techniques from pseudo-differential operator theory.

Problem: *The limits (1) and (2) do not require derivatives. Can we relax the regularity of f and g?*

We start with (2): Consider the big Hankel operator

$$H_f^t = (I - P^t)M_f : H_t^2 \to (H_t^2)^\perp.$$

Standard identity:

$$T_f^t T_g^t - T_{fg}^t = -(H_{\bar{f}}^t)^* H_g^t.$$

Corollary

Let f and g be symbols s.t. $t \mapsto \|H_g^t\|_t$ is bounded as $t \to 0$ and

$$\lim_{t\to 0} \|H_{\bar{f}}^t\|_t = 0.$$

Then (2) holds, i.e. $\lim_{t\to 0} \|T_f^t T_g^t - T_{fg}^t\|_t = 0$.

Consequence: We need to estimate the norm of H_f^t as $t \to 0$.

Theorem D (W. -B., L. Coburn, R. Hagger)

Let $f \in \mathrm{BMO}^2(\mathbb{C}^n)$ and assume that there is $c > 0$ such that for all $t \in (\alpha, \beta)$ where $0 \leq \alpha < \beta$:

$$\|f - \tilde{f}^{(2t)}\|_\infty \leq c. \qquad (*)$$

Then there is $C > 0$ independent of t such that for all $t \in (\alpha, \beta)$

$$\|H_f^t\|_t \leq C\Big\{\sqrt{\|f\|_{\mathrm{BMO}^2_{2t}}} + \|f\|_{\mathrm{BMO}^2_{2t}}\Big\}. \qquad (**)$$

Remark: The condition $(*)$ holds for $f \in L^\infty(\mathbb{C}^n)$ or $f \in \mathrm{UC}(\mathbb{C}^n)$.

Consequence: A problem on operators is reduced to a problem on functions. We need to decide for which symbols with $(*)$ we have

$$\lim_{t\to 0} \|f\|_{\mathrm{BMO}^2_{2t}} = 0.$$

Theorem E (W. -B., L. Coburn, R. Hagger)

Let $f \in \mathrm{UC}(\mathbb{C}^n)$. Then

- $\tilde{f}^{(t)} \to f$ uniformly on $\mathbb{C}^n$ as $t \to 0$ (approximate UC by Lip)
- $\|f\|_{\mathrm{BMO}_t^2} \to 0$ as $t \to 0$.

In particular: (by Theorem D):

- $\lim_{t\to 0} \|H_f^t\|_t = 0$.

In particular, (by the Corollary)

For all $g \in L^\infty(\mathbb{C}^n)$ or $g \in \mathrm{UC}(\mathbb{C}^n)$ we have (2), i.e.

$$\lim_{t\to 0} \| T_f^t T_g^t - T_{fg}^t \|_t = 0.$$

Note: The Toeplitz operators T_f^t or T_g^t above may be unbounded. However, the semi-commutator

$$T_f^t T_g^t - T_{fg}^t$$

necessarily is bounded for all $t > 0$.

Remarks

- Theorem E also holds for **bounded** operator symbols in

  $\mathrm{VMO}(\mathbb{C}^n)$ = *function of vanishing mean oscillation*.

  Recall: $f \in \mathrm{VMO}(\mathbb{C}^n)$ iff for all cubes E in $\mathbb{C}^n$:

$$\lim_{a\to 0} \left\{ \frac{1}{|E|} \int_E |f - f_E| \; : \; |E| \leq a \right\} = 0, \quad \text{with } f_E = \frac{1}{|E|} \int_E f.$$

- The space $\mathrm{VMO}(\mathbb{C}^n)$ contains non-continuous functions.

Remarks

- THEOREM E **does not hold** i.g. for symbols of high oscillation, e.g.

$$\lim_{t\to 0}(T_g^t T_f^t - T_{gf}^t)1 = -1$$

if $f, g \in L^\infty(\mathbb{C}^n)$ are chosen as follows:

$$f(z) = \overline{g}(z) = \begin{cases} 1 & \text{if } z = 0, \\ e^{\frac{i}{|z|^2}} & \text{if } z \neq 0 \end{cases}$$

- The asymptotic relation (1) is always true, i.e. without any further conditions on f:

$$\lim_{t\to 0} \|T_f^t\|_t = \|f\|_\infty \quad \text{for all} \quad f \in L^\infty(\mathbb{C}^n).$$

Fock Quantization Algebras

There is a kind of converse of THEOREM E, which I describe next:

Consider direct integrals and operators:

$$L^2 = \int_{\mathbb{R}_+}^{\oplus} L^2(\mathbb{C}^n, \mu_t)$$

$$X = \oplus_{t>0} X^{(t)} \quad \text{where} \quad X^{(t)} \in \mathcal{L}\big(L^2(\mathbb{C}^n, \mu_t)\big)$$

with norm

$$\|X\| = \sup_{t>0} \|X^{(t)}\| < \infty.$$

Definition

The algebra of such operators is denoted by $\mathrm{Op}(L^2)$. Define

$$\mathcal{I} := \Big\{ X \in \mathrm{Op}(L^2) \;:\; \lim_{t\to 0} \|X^{(t)}\| = 0 \Big\},$$

which is a closed two-sided ideal in $\mathrm{Op}(L^2)$.

Fock Quantization Algebras

Similarly: $H^2 = \int_{\mathbb{R}_+}^{\oplus} H_t^2$ and $\mathrm{Op}(H^2)$.

Example

Let $f \in L^\infty(\mathbb{C}^n)$, then:

$$\mathbf{T_f} = \oplus_{t>0} T_f^t \in \mathrm{Op}(H^2) \quad \textit{and} \quad \mathbf{H_f} := \oplus_{t>0} H_f^t \in \mathrm{Op}(L^2),$$

Lemma: The set

$$\mathcal{A} = \Big\{ f \in L^\infty(\mathbb{C}^n) \; : \; \mathbf{T_f T_g} - \mathbf{T_{fg}}, \mathbf{T_g T_f} - \mathbf{T_{fg}} \in \mathcal{I}, \; \forall \, g \in L^\infty(\mathbb{C}^n) \Big\}$$

is a closed, conjugate-closed subalgebra of $L^\infty(\mathbb{C}^n)$ and coincides with

$$\mathcal{A} = \Big\{ f \in L^\infty(\mathbb{C}^n) \; : \; \mathbf{H_f}, \mathbf{H_{\bar{f}}} \in \mathcal{I} \Big\}.$$

Fock Quantization algebras

Consider a second set of bounded functions:

$$\mathcal{B} := \Big\{ f \in L^\infty(\mathbb{C}^n) \; : \; \lim_{t \to 0} \|f\|_{\mathrm{BMO}_t^2} = 0 \Big\}.$$

Then one can show that:

Theorem (W.B., L. Coburn, R. Hagger)

$\mathcal{A} = \mathcal{B} = \mathrm{VMO}(\mathbb{C}^n) \cap L^\infty(\mathbb{C}^n)$. In particular, $\mathcal{B}$ is a C^* function algebra.

Remarks: In the above sense $\mathrm{VMO} \cap L^\infty(\mathbb{C}^n)$ is the largest C^* symbol algebra for which the quantization estimates (2) hold.

2. Application: Toeplitz algebras over the two ball

Let $\mathbb{B}^n \subset \mathbb{C}^n$ denote the open unit ball. Let $\lambda > -1$ (weight).

$$\begin{aligned} dv_\lambda(z) &= c_\lambda(1-|z|^2)^\lambda dv(z) \\ &= \textit{weighted (probability) measure.} \end{aligned}$$

Definition (weighted Bergman space)

$$\mathcal{A}^2_\lambda(\mathbb{B}^n) := \mathcal{O}(\mathbb{B}^n) \cap L^2(\mathbb{B}^n, dv_\lambda).$$

Consider the orthogonal projection:

$$P_\lambda : L^2(\mathbb{B}^n, dv_\lambda) \to \mathcal{A}^2_\lambda(\mathbb{B}^n).$$

Definition (Toeplitz operator)

Let $f \in L^\infty(\mathbb{B}^n)$. The Toeplitz operator with symbol f is defined by

$$T_f^\lambda : \mathcal{A}^2_\lambda(\mathbb{B}^n) \to \mathcal{A}^2_\lambda(\mathbb{B}^n) : g \mapsto P_\lambda(f \cdot g).$$

Quantization estimates on the ball

We equipp $\mathbb{B}^n$ with the Bergman metric distance β and consider the corresponding function spaces

$$\mathrm{UC}(\mathbb{B}^n) \qquad \textit{and} \qquad \mathrm{VMO}(\mathbb{B}^n).$$

Theorem, (W.B., R. Hagger, N. Vasilevski)

Let $f \in \mathrm{UC}(\mathbb{B}^n)$, (**or** $f \in \mathrm{VMO}(\mathbb{B}^n)$), then

$$\lim_{\lambda\to\infty} \left\| T_f^\lambda T_g^\lambda - T_{fg}^\lambda \right\|_\lambda = 0$$

for all $g \in L^\infty(\mathbb{B}^n)$ **or** all $g \in \mathrm{UC}(\mathbb{B}^n)$.

Remark: *Similarly for any bounded symmetric domains.*

Toeplitz algebras over the two-ball

From now on: *Put* $n = 2$.

With weight $\lambda > -1$ we consider the standard ONB of $\mathcal{A}^2_\lambda(\mathbb{B}^2)$:

$$\mathcal{B}_\lambda := \left\{ \frac{z^\alpha}{\|z^\alpha\|_\lambda} \; : \; \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+ \right\}.$$

Consider a sequence of Hilbert subspaces of $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ defined by:

$$H_{\alpha_2} := \overline{\text{span}}\left\{ \frac{z^\alpha}{\|z^\alpha\|_\lambda} \; : \; \alpha = (\alpha_1, \alpha_2), \; \alpha_1 \in \mathbb{Z}_+, \;\; \alpha_2 \in \mathbb{Z}_+ \textit{ fixed} \right\}.$$

Decomposition

One obtains an orthogonal decomposition of the Bergman space:

$$\mathcal{A}^2_\lambda(\mathbb{B}^2) := \bigoplus_{\alpha_2 \in \mathbb{Z}_+} H_{\alpha_2}.$$

Reduction of dimension

For each $\alpha_2 \in \mathbb{Z}_+$ there is a well-defined and unitary map:

$$u_{\alpha_2} : H_{\alpha_2} \to \mathcal{A}^2_{\alpha_2+\lambda+1}(\mathbb{D}) : f(z_1) \cdot \frac{z_2^{\alpha_2}}{\|z_2^{\alpha_2}\|_{\lambda+1}} \mapsto f(z_1).$$

Note: $\frac{z^\alpha}{\|z^\alpha\|_\lambda} = \frac{z_1^{\alpha_1}}{\|z^{\alpha_1}\|_{\lambda+\alpha_2+1}} \cdot \frac{z_2^{\alpha_2}}{\|z^{\alpha_2}\|_{\lambda+1}}$, where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+$.

Proposition

The operator U below is an isometric isomorphism:

$$U = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} u_{\alpha_2} : \mathcal{A}^2_\lambda(\mathbb{B}^2) \to \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \mathcal{A}^2_{\alpha_2+\lambda+1}(\mathbb{D}).$$

Question:

Which Toeplitz operators on $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ leave - after conjugation with U - the space $\mathcal{A}^2_{\alpha_2+\lambda+1}(\mathbb{D})$ invariant for all $\alpha_2 \in \mathbb{Z}_+$?

Proposition

Let $a \in L^\infty(\mathbb{D})$ and $b \in L^\infty(0,1)$ and put:

$$f_{ab}(z) := a(z_1) \cdot b\left(\frac{|z_2|}{\sqrt{1-|z_2|^2}}\right).$$

The Toeplitz operator $\mathbf{T}^\lambda_{f_{ab}}$ on $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ decomposes as

$$U\mathbf{T}^\lambda_{f_{ab}}U^* = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \gamma^\lambda_b(\alpha_2) T^{\alpha_2+\lambda+1}_a,$$

(a) $T^{\alpha_2+\lambda+1}_a$ = Toeplitz operator acting on $\mathcal{A}^2_{\alpha_2+\lambda+1}(\mathbb{D})$.

(b) Moreover, for all $\alpha_2 \in \mathbb{Z}_+$:

$$\gamma^\lambda_b(\alpha_2) = \frac{\Gamma(\alpha_2+\lambda+2)}{\Gamma(\alpha_2+1)\Gamma(\lambda+1)} \int_0^1 b(\sqrt{s})s^{\alpha_2}(1-s)^\lambda ds.$$

Goal: *Study C^* algebras generated by Toeplitz operators which leave the above decomposition of $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ invariant.*

Construction of operator algebras:

Chose subclasses $\mathcal{S}_1 \subset L^\infty(\mathbb{D})$ and $\mathcal{S}_2 \subset L^\infty(0,1)$ and consider the C^* algebra:

$$\mathcal{T}^{\mathbb{B}^2}_\lambda(\mathcal{S}_1, \mathcal{S}_2) := C^*\left\{\mathbf{T}^\lambda_{f_{ab}} \; : \; a \in \mathcal{S}_1 \quad \textit{and} \quad b \in \mathcal{S}_2\right\}.$$

Notation: Let $\mathcal{S}$ denote a set of bounded operators. Put:

$$C^*(\mathcal{S}) := C^* \textit{ algebra generated by the operators in } \mathcal{S}.$$

Example: Special (commutative) case

$$\mathcal{T}^{\mathbb{B}^2}_\lambda(\{1\}, L^\infty(0,1)) \cong \mathrm{SO}(\mathbb{Z}_+),$$

where

$$\mathrm{SO}(\mathbb{Z}_+) = \left\{(a_j)_{j\in\mathbb{Z}_+} \; : \; \lim_{\frac{j+1}{k+1}\to 1} |a_j - a_k| = 0\right\}.$$

A non-commutative case

Consider the C^* algebra:

$$\mathcal{T}_\lambda^{\mathbb{B}^2}\big(C(\overline{\mathbb{D}}), \{1\}\big) = C^*\big\{\mathbf{T}_{f_a}^\lambda \;:\; a \in C(\overline{\mathbb{D}})\big\}.$$

Theorem (W. B., N. Vasilevski, 2017)

Each $\mathbf{T}^\lambda$ in $\mathcal{T}_\lambda^{\mathbb{B}^2}\big(C(\overline{\mathbb{D}}), \{1\}\big)$ has a unique sum decomposition:

$$U\mathbf{T}^\lambda U^* = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \big(T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}\big), \qquad (*)$$

where $a \in C(\overline{\mathbb{D}})$ and K^{α_2} is compact with norm convergence:

$$\mathcal{K}\big(\mathcal{A}^2_{\alpha_2+\lambda+1}(\mathbb{D})\big) \ni K^{\alpha_2} \to 0 \quad \text{as} \quad \alpha_2 \to \infty.$$

Question: *How can we recover the symbol a of $\mathbf{T}^\lambda$ in $(*)$?*

The symbol map "via Quantization"

Theorem

The map $\rho : \mathcal{T}_\lambda^{\mathbb{B}^2}\big(C(\overline{\mathbb{D}}), \{1\}\big) \to C(\overline{\mathbb{D}})$:

$$\rho : \mathbf{T}^\lambda \mapsto U\mathbf{T}^\lambda U^* = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2+\lambda+1}$$

$$\mapsto \lim_{\alpha_2 \to \infty} \underbrace{B_{\alpha_2+\lambda+1}\big(T^{\alpha_2+\lambda+1}\big)}_{\text{Berezin transform}} \in C(\overline{\mathbb{D}})$$

is a continuous and surjective $*$-homomorphism of C^* algebras.

Answer: *The homomorphism ρ recovers the function*

$$a = \rho(\mathbf{T}^\lambda) \in C(\overline{\mathbb{D}})$$

in the representation $(*)$ of $\mathbf{T}^\lambda$:

$$U\mathbf{T}^\lambda U^* = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \big(T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}\big). \qquad (*)$$

Fredholm property and index

Theorem

Let $\mathbf{T}^\lambda \in \mathcal{T}_\lambda^{\mathbb{B}^2}(C(\overline{\mathbb{D}}), \{1\})$*. The following are equivalent:*

(1) $\mathbf{T}^\lambda$ *is a Fredholm operator,*

(2) $\rho(\mathbf{T}^\lambda) \in C(\overline{\mathbb{D}})$ *is invertible, i.e. pointwise non-vanishing.*

If (1) *and* (2) *are true then*

$$\mathrm{Ind}(\mathbf{T}^\lambda) = 0.$$

The essential spectrum of $\mathbf{T}^\lambda$ *is given by:*

$$\sigma_{\mathrm{ess}}(\mathbf{T}^\lambda) = \mathrm{Range}\ \rho(\mathbf{T}^\lambda).$$

Irreducible representations

Theorem (W.B., N. Vasilevski, 2017)

A complete list of irreducible representations of the C^* *algebra*

$$\mathcal{T}_\lambda^{\mathbb{B}^2}(C(\overline{\mathbb{D}}), \{1\})$$

is given as follows:

(i) *infinite dimensional repr. (non-equivalent for different* α_2*):*

$$\iota_{\alpha_2} : \mathbf{T}^\lambda \mapsto U\mathbf{T}^\lambda U^* = \bigoplus_{\beta_2 \in \mathbb{Z}_+} T_a^{\beta_2+\lambda+1} + K^{\beta_2} \mapsto T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}.$$

(ii) *The one-dimensional representations: Let* $t \in \overline{\mathbb{D}}$*, then put:*

$$\pi_t(\mathbf{T}^\lambda) = \rho(\mathbf{T}^\lambda)(t) \in \mathbb{C}.$$

Further problems:

(a) How does the above analysis generalizes to the larger algebra

$$\mathcal{A}_\lambda := \mathcal{T}_\lambda^{\mathbb{B}^2}(C(\overline{\mathbb{D}}), L^\infty(0,1))?$$

Some new effects:

- Representations of elements in the form

$$\bigoplus_{\alpha_2 \in \mathbb{Z}_+} \left(T^\lambda_{c(z_1,\alpha_2)} + K^{\alpha_2} \right) \in \mathcal{A}_\lambda$$

  are not unique anymore.
- $\mathcal{A}_\lambda$ contains Toeplitz operators with non-zero index.
- index formulas exist (W.B., R. Hagger, N. Vasilevski).

Further problems:

(b) What happens if we further enlarge the algebra by replacing $C(\overline{\mathbb{D}})$ with a bigger function algebras $\mathcal{S}_a$, e.g.

$$\mathcal{S} = \mathrm{VO}_\partial(\mathbb{D}) = \text{"}\textit{vanishing oscillation at the boundary}\text{"},$$
$$\mathcal{S} = \mathrm{BUC}?$$

Some results:

Based on the quantization results of the first part and compactness of semi-commutators we treat the algebras

$$\mathcal{T}_\lambda^{\mathbb{B}^2}(\{1\}, \mathrm{VO}_\partial(\mathbb{D})) \quad \textit{and even} \quad \mathcal{T}_\lambda^{\mathbb{B}^2}(L^\infty_{\text{k-qr}}(\mathbb{B}^\ell), \mathrm{VO}_\partial(\mathbb{B}^{n-\ell}))$$

Theorem

Let $c(\rho, z'') \in \left(\mathrm{SO}(\mathbb{Z}_+^m) \otimes \mathrm{VO}_\partial(\mathbb{B}^{n-\ell})\right) \otimes \mathrm{Mat}_p(\mathbb{C})$. *Then the Toeplitz operator*

$$\mathbf{T}^\lambda \asymp \bigoplus_{\rho \in \mathbb{Z}_+^m} I \otimes (T_c^{\lambda+|\rho|+\ell} + K_\rho) \in \mathcal{T}_\lambda(L^\infty_{k\text{-}qr}, \mathrm{VO}_\partial(\mathbb{B}^{n-\ell})) \otimes \mathrm{Mat}_p(\mathbb{C})$$

is Fredholm if and only if the restriction of the matrix $c(\eta)$ *onto*

$$(M_\infty \times M(\mathrm{VO})) \cup (\mathbb{Z}_+^m \times M_\partial)$$

is invertible. The essential spectrum and index are given by

$$\sigma_{\mathrm{ess}}(\mathbf{T}^\lambda) = \mathrm{Range}\ \det c|_{M_\infty \times M(\mathrm{VO}) \cup \mathbb{Z}_+^m \times M_\partial}.$$

$$\mathrm{Ind}(\mathbf{T}^\lambda) = \sum_{\rho \in \mathbb{Z}_+^m} \dim \mathcal{H}_\rho \times \mathrm{Ind}\left(T_{c_{s_\rho}}^{\lambda+|\rho|+\ell}\right).$$

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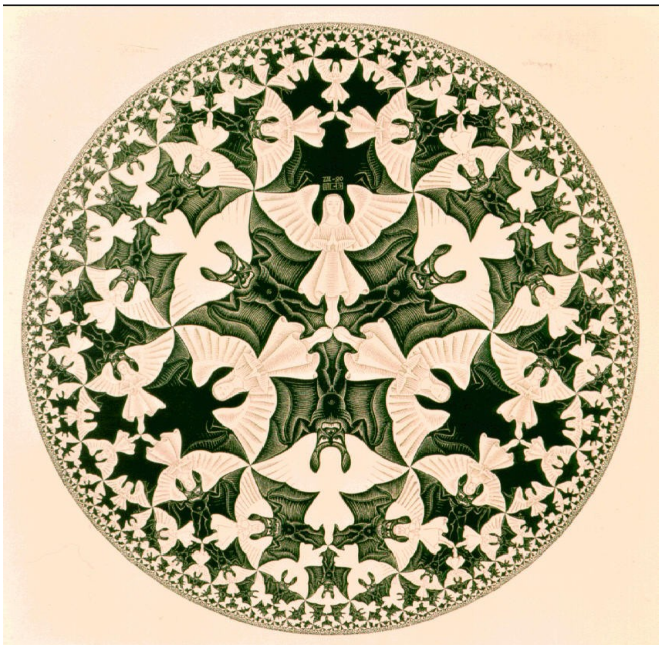
Figure: M.C. Escher: Circle Limit IV



Thank you for your attention!