Loop groups, coadjoint orbits, and localization formulas

Jean-Michel Bismut

Institut de Mathématique d'Orsay

Cologne, July 25$^{\text{th}}$2023

QUANTIZATION IN GEOMETRY

A compact Lie group

A compact Lie group

- G compact connected Lie group, $\mathfrak{g}$ Lie algebra, B G-invariant scalar product on $\mathfrak{g}$.

A compact Lie group

- G compact connected Lie group, $\mathfrak{g}$ Lie algebra, B G-invariant scalar product on $\mathfrak{g}$.
- S^1 acts on LG, $\widehat{L}G = S^1 \rtimes LG$.

A compact Lie group

- G compact connected Lie group, $\mathfrak{g}$ Lie algebra, B G-invariant scalar product on $\mathfrak{g}$.
- S^1 acts on LG, $\widehat{L}G = S^1 \rtimes LG$.
- Lie algebra $\widehat{L}\mathfrak{g} = \left\{a\frac{d}{ds} + A_s\right\}, a \in \mathbf{R}, A_\cdot \in L\mathfrak{g}$.

The coadjoint orbits

The coadjoint orbits

- In the sequel, we take $a = 1$, $\left\{\frac{d}{ds} + A_s\right\}$ = connections on the trivial G-bundle over S^1.

The coadjoint orbits

- In the sequel, we take $a = 1$, $\left\{\frac{d}{ds} + A_s\right\}$ = connections on the trivial G-bundle over S^1.
- Two connections gauge equivalent $\Leftrightarrow$ they have conjugate holonomies.

The coadjoint orbits

- In the sequel, we take $a = 1$, $\left\{ \frac{d}{ds} + A_s \right\}$ = connections on the trivial G-bundle over S^1.
- Two connections gauge equivalent $\Leftrightarrow$ they have conjugate holonomies.
- If $g \in G$, $O_g \subset G$ adjoint orbit in G.

The coadjoint orbits

- In the sequel, we take $a = 1$, $\left\{\frac{d}{ds} + A_s\right\}$ = connections on the trivial G-bundle over S^1.
- Two connections gauge equivalent $\Leftrightarrow$ they have conjugate holonomies.
- If $g \in G$, $O_g \subset G$ adjoint orbit in G.
- Coadjoint orbit $\mathcal{O}_g \subset \widehat{L\mathfrak{g}}$ = smooth paths in G connecting 1 and O_g via parallel transport.

The coadjoint orbits

- In the sequel, we take $a = 1$, $\left\{\frac{d}{ds} + A_s\right\}$ = connections on the trivial G-bundle over S^1.
- Two connections gauge equivalent $\Leftrightarrow$ they have conjugate holonomies.
- If $g \in G$, $O_g \subset G$ adjoint orbit in G.
- Coadjoint orbit $\mathcal{O}_g \subset \widehat{L\mathfrak{g}}$ = smooth paths in G connecting 1 and O_g via parallel transport.
- $\frac{d}{ds} + A \leftrightarrow \frac{dg}{ds} + Ag = 0$.

The symplectic structure

The symplectic structure

- $\mathcal{O}_g$ inherits canonical symplectic structure.

The symplectic structure

- $\mathcal{O}_g$ inherits canonical symplectic structure.
- If $\alpha, \beta \in L\mathfrak{g}$, $\omega_A\left(D_A\alpha, D_A\beta\right) = \int_{S^1} \langle D_A\alpha, \beta\rangle$.

The symplectic structure

- $\mathcal{O}_g$ inherits canonical symplectic structure.
- If $\alpha, \beta \in L\mathfrak{g}$, $\omega_A(D_A\alpha, D_A\beta) = \int_{S^1} \langle D_A\alpha, \beta\rangle$.
- S^1 acts symplectically on $\mathcal{O}_g$ and the corresponding Hamiltonian is $E(A) = \frac{1}{2}\int_{S^1} |A|^2 = \frac{1}{2}\int_{S^1} |\dot{g}|^2 \, ds$.

The symplectic structure

- $\mathcal{O}_g$ inherits canonical symplectic structure.
- If $\alpha, \beta \in L\mathfrak{g}$, $\omega_A \left(D_A \alpha, D_A \beta \right) = \int_{S^1} \langle D_A \alpha, \beta \rangle$.
- S^1 acts symplectically on $\mathcal{O}_g$ and the corresponding Hamiltonian is $E\left(A\right) = \frac{1}{2} \int_{S^1} \left|A\right|^2 = \frac{1}{2} \int_{S^1} \left|\dot{g}\right|^2 ds$.
- $\widetilde{LG}$ central extension of $\widehat{LG}$.

The symplectic structure

- $\mathcal{O}_g$ inherits canonical symplectic structure.
- If $\alpha, \beta \in L\mathfrak{g}$, $\omega_A\left(D_A\alpha, D_A\beta\right) = \int_{S^1} \left\langle D_A\alpha, \beta\right\rangle$.
- S^1 acts symplectically on $\mathcal{O}_g$ and the corresponding Hamiltonian is $E\left(A\right) = \frac{1}{2}\int_{S^1} \left|A\right|^2 = \frac{1}{2}\int_{S^1} \left|\dot{g}\right|^2 ds$.
- $\widetilde{L}G$ central extension of $\widehat{LG}$.
- Full coadjoint orbit $= \left\{\frac{d}{ds} + A_s, E\left(A\right)\right\}$.

The heat kernel on G

The heat kernel on G

- $p_t(g)$ smooth kernel for $\exp\left(t\Delta^G/2\right)$.

The heat kernel on G

- $p_t(g)$ smooth kernel for $\exp\left(t\Delta^G/2\right)$.
- $p_t(g)$ Ad-invariant function.

The heat kernel on G

- $p_t(g)$ smooth kernel for $\exp\left(t\Delta^G/2\right)$.
- $p_t(g)$ Ad-invariant function.
- $p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\int_{S^1} |\dot{g}|^2 \, ds/2t\right) \frac{\mathcal{D}g}{t^{\infty/2}}$ path integral.

The heat kernel on G

- $p_t(g)$ smooth kernel for $\exp\left(t\Delta^G/2\right)$.
- $p_t(g)$ Ad-invariant function.
- $p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\int_{S^1} |\dot{g}|^2 ds/2t\right) \frac{\mathcal{D}g}{t^{\infty/2}}$ path integral.
- Could it be that $p_t(g)$ is a Duistermaat-Heckman, Berline-Vergne integral

$$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right).$$

The heat kernel on G

- $p_t(g)$ smooth kernel for $\exp\left(t\Delta^G/2\right)$.
- $p_t(g)$ Ad-invariant function.
- $p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\int_{S^1} |\dot{g}|^2 ds/2t\right) \frac{\mathcal{D}g}{t^{\infty/2}}$ path integral.
- Could it be that $p_t(g)$ is a Duistermaat-Heckman, Berline-Vergne integral

$$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right).$$

- Are the corresponding localization formulas correct?

The approach by I. Frenkel via geometric quantization

The approach by I. Frenkel via geometric quantization

- Class of representations of $\widetilde{LG}$ with positive energy and geometric quantization of $\mathcal{O}_g$.

The approach by I. Frenkel via geometric quantization

- Class of representations of $\widetilde{LG}$ with positive energy and geometric quantization of $\mathcal{O}_g$.
- I. Frenkel proved that: $p_t(g)$ numerator of Kac character's formula for $\widetilde{LG}$.

The approach by I. Frenkel via geometric quantization

- Class of representations of $\widetilde{LG}$ with positive energy and geometric quantization of $\mathcal{O}_g$.
- I. Frenkel proved that: $p_t(g)$ numerator of Kac character's formula for $\widetilde{LG}$.
- Proof: Kac's ch. formula with coroot lattice,

The approach by I. Frenkel via geometric quantization

- Class of representations of $\widetilde{LG}$ with positive energy and geometric quantization of $\mathcal{O}_g$.
- I. Frenkel proved that: $p_t(g)$ numerator of Kac character's formula for $\widetilde{LG}$.
- Proof: Kac's ch. formula with coroot lattice,
- ... and Fourier analysis to pass to the root lattice and obtain $p_t(g)$ via spectral theory.

The approach by I. Frenkel via geometric quantization

- Class of representations of $\widetilde{LG}$ with positive energy and geometric quantization of $\mathcal{O}_g$.
- I. Frenkel proved that: $p_t(g)$ numerator of Kac character's formula for $\widetilde{LG}$.
- Proof: Kac's ch. formula with coroot lattice,
- …and Fourier analysis to pass to the root lattice and obtain $p_t(g)$ via spectral theory.
- Application of Kirillov to Lefschetz principle gives a formal understanding of the appearance of $p_t(g)$ in the numerator.

The formula by Frenkel: the spectral side

The formula by Frenkel: the spectral side

By spectral theory,

$$\int_G p_t\left(g_1 g g_2^{-1} g^{-1}\right) dg$$
$$= \sum_{\lambda \in \underbrace{P_{++}}_{\text{positive roots}}} \chi_\lambda\left(g_1\right) \chi_\lambda\left(g_2^{-1}\right) \exp\left(-\frac{t}{2}\left(\left|\lambda+\rho\right|^2 - \left|\rho\right|^2\right)\right).$$

The formula by Frenkel: the geometric side

The formula by Frenkel: the geometric side

Via Poisson formula,

$$e^{-4\pi^2|\rho|^2 t/2} \int_G p_t \left(g e^{t_1} g^{-1} e^{-t_2} \right) dg = \frac{\mathrm{Vol}\,(T)}{(2\pi t)^{m/2}} \sigma^{-1}(t_1)\, \sigma^{-1}(-t_2) \sum_{(w,\gamma) \in W \times \underbrace{\overline{CR}}_{\text{coroot lattice}}} \epsilon_w \exp\left(-\frac{|t_2 - w t_1 + \gamma|^2}{2t} \right).$$

The formula by Frenkel: the geometric side

Via Poisson formula,

$$e^{-4\pi^2|\rho|^2 t/2}\int_G p_t\left(ge^{t_1}g^{-1}e^{-t_2}\right)dg=\frac{\mathrm{Vol}\left(T\right)}{\left(2\pi t\right)^{m/2}}\sigma^{-1}\left(t_1\right)\sigma^{-1}\left(-t_2\right)$$

$$\sum_{(w,\gamma)\in W\times\underbrace{\overline{CR}}_{\text{coroot lattice}}}\epsilon_w\exp\left(-\frac{\left|t_2-wt_1+\gamma\right|^2}{2t}\right).$$

•This is the numerator of Kac's character formula for $\widetilde{LG}$.

The formula by Frenkel: the geometric side

Via Poisson formula,

$$e^{-4\pi^2|\rho|^2 t/2}\int_G p_t\left(ge^{t_1}g^{-1}e^{-t_2}\right)dg = \frac{\mathrm{Vol}\,(T)}{(2\pi t)^{m/2}}\sigma^{-1}(t_1)\,\sigma^{-1}(-t_2)$$

$$\sum_{(w,\gamma)\in W\times \underbrace{\overline{CR}}_{\text{coroot lattice}}} \epsilon_w \exp\left(-\frac{|t_2 - wt_1+\gamma|^2}{2t}\right).$$

- This is the numerator of Kac's character formula for $\widetilde{LG}$.
- Interpreted by Frenkel as Kirillov-Lefschetz.

The formula by Frenkel: the geometric side

Via Poisson formula,

$$e^{-4\pi^2|\rho|^2 t/2}\int_G p_t\left(g e^{t_1} g^{-1} e^{-t_2}\right)dg = \frac{\mathrm{Vol}\,(T)}{(2\pi t)^{m/2}}\sigma^{-1}\left(t_1\right)\sigma^{-1}\left(-t_2\right)$$

$$\sum_{(w,\gamma)\in W\times \underbrace{\overline{CR}}_{\text{coroot lattice}}} \epsilon_w \exp\left(-\frac{\left|t_2 - w t_1 + \gamma\right|^2}{2t}\right).$$

- This is the numerator of Kac's character formula for $\widetilde{LG}$.
- Interpreted by Frenkel as Kirillov-Lefschetz.
- View this identity as direct consequence of DH, BV?

The approach by Atiyah

The approach by Atiyah

- Atiyah explained the formal relation of index theory for the Dirac operator on X to equivariant cohomology of LX.

The approach by Atiyah

- Atiyah explained the formal relation of index theory for the Dirac operator on X to equivariant cohomology of LX.
- Atiyah takes for granted the formal formula

$$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\ldots.$$

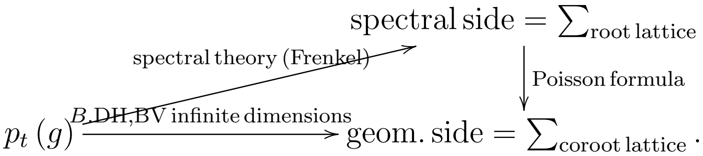
The approach by Atiyah

- Atiyah explained the formal relation of index theory for the Dirac operator on X to equivariant cohomology of LX.
- Atiyah takes for granted the formal formula

$$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\ldots.$$

- He guesses that application of Duistermaat-Heckman, Berline-Vergne to this formula gives a correct formula for $p_t(g)$!

A diagram


spectral side = $\sum_{\text{root lattice}}$
spectral theory (Frenkel)
Poisson formula
B,DH,BV infinite dimensions
$p_t(g)$
geom. side = $\sum_{\text{coroot lattice}}$.

Proof of DH, BV

Proof of DH, BV

- X compact Riemannian manifold, K Killing vector field, $X_K = (K = 0)$.

Proof of DH, BV

- X compact Riemannian manifold, K Killing vector field, $X_K = (K = 0)$.
- $d_K = d + i_K$, $d_K^2 = L_K$.

Proof of DH, BV

- X compact Riemannian manifold, K Killing vector field, $X_K = (K = 0)$.
- $d_K = d + i_K$, $d_K^2 = L_K$.

Theorem (BV 83)

If μ form such that $d_K \mu = 0$, then

$$\int_X \mu = \int_{X_K} \frac{\mu}{e_K\left(N_{X_K/X}\right)}.$$

A Gaussian proof of DH, BV (B86)

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.
- $d_K K' = |K|^2 + dK'$, $d_K d_K K' = 0$.

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.
- $d_K K' = |K|^2 + dK'$, $d_K d_K K' = 0$.
- $\alpha_t = \exp\left(-d_K K'/2t\right)$, $d_K \alpha_t = 0$, $\frac{\partial}{\partial t}\alpha_t = \frac{1}{2t^2} d_K \left[K' \alpha_t\right]$.

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.
- $d_K K' = |K|^2 + dK'$, $d_K d_K K' = 0$.
- $\alpha_t = \exp(-d_K K'/2t)$, $d_K \alpha_t = 0$, $\frac{\partial}{\partial t}\alpha_t = \frac{1}{2t^2} d_K [K'\alpha_t]$.
- $\int_X \mu = \int_X \alpha_t \mu$.

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.
- $d_K K' = |K|^2 + dK'$, $d_K d_K K' = 0$.
- $\alpha_t = \exp(-d_K K'/2t)$, $d_K \alpha_t = 0$, $\frac{\partial}{\partial t}\alpha_t = \frac{1}{2t^2} d_K [K'\alpha_t]$.
- $\int_X \mu = \int_X \alpha_t \mu$.
- $$\text{Proof} : \frac{\partial}{\partial t}\int_X \alpha_t \mu = \frac{1}{2t^2}\int_X d_K [K'\alpha_t \mu] = 0.$$

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.
- $d_K K' = |K|^2 + dK'$, $d_K d_K K' = 0$.
- $\alpha_t = \exp(-d_K K'/2t)$, $d_K \alpha_t = 0$, $\frac{\partial}{\partial t}\alpha_t = \frac{1}{2t^2} d_K [K'\alpha_t]$.
- $\int_X \mu = \int_X \alpha_t \mu$.
- $$\text{Proof} : \frac{\partial}{\partial t}\int_X \alpha_t \mu = \frac{1}{2t^2}\int_X d_K [K'\alpha_t \mu] = 0.$$
- B86 As $t \to 0$, $\alpha_t \to \frac{\delta_{X_K}}{e_K\left(N_{X_K/X}, \nabla^{N_{X_K/X}}\right)}$ as a current.

A Gaussian proof of DH, BV (B86)

- K' 1-form dual to K, $L_K K' = 0$.
- $d_K K' = |K|^2 + dK'$, $d_K d_K K' = 0$.
- $\alpha_t = \exp\left(-d_K K'/2t\right)$, $d_K \alpha_t = 0$, $\frac{\partial}{\partial t}\alpha_t = \frac{1}{2t^2} d_K \left[K'\alpha_t\right]$.
- $\int_X \mu = \int_X \alpha_t \mu$.
- $$\text{Proof}: \frac{\partial}{\partial t}\int_X \alpha_t \mu = \frac{1}{2t^2}\int_X d_K \left[K'\alpha_t \mu\right] = 0.$$
- B86 As $t \to 0$, $\alpha_t \to \dfrac{\delta_{X_K}}{e_K\left(N_{X_K/X}, \nabla^{N_{X_K/X}}\right)}$ as a current.
- ‘Explains’ fantastic cancellations in local index theory.

Formal proof of DH, BV on $\mathcal{O}_g$

Formal proof of DH, BV on $\mathcal{O}_g$

- $$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\dots$$

Formal proof of DH, BV on $\mathcal{O}_g$

- $$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\ldots$$
- L_2 metric on $\mathcal{O}_g$: $\left|\delta A_s\right|^2 = \int_{S^1} \left|\delta A\right|^2 ds$.

Formal proof of DH, BV on $\mathcal{O}_g$

- $$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\ldots$$
- L_2 metric on $\mathcal{O}_g$: $\left|\delta A_s\right|^2 = \int_{S^1} \left|\delta A\right|^2 ds$.
- S^1 acts isometrically on $\mathcal{O}_g$: $A_\cdot \to A_{\cdot+h}$.
- $K = \dot{A}$, $\left|K\right|^2 = \int_{S^1} |\dot{A}_s|^2 ds$.

Formal proof of DH, BV on $\mathcal{O}_g$

- $$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\dots$$
- L_2 metric on $\mathcal{O}_g$: $\left|\delta A_s\right|^2 = \int_{S^1} \left|\delta A\right|^2 ds$.
- S^1 acts isometrically on $\mathcal{O}_g$: $A_\cdot \to A_{\cdot+h}$.
- $K = \dot{A}$, $\left|K\right|^2 = \int_{S^1} |\dot{A}_s|^2 ds$.
- Replace DH, BV integral for $p_t(g)$ by $p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right) \exp\left(-b^4 d_K K'/2\right)$.

Formal proof of DH, BV on $\mathcal{O}_g$

- $$p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right)\dots$$
- L_2 metric on $\mathcal{O}_g$: $\left|\delta A_s\right|^2 = \int_{S^1} \left|\delta A\right|^2 ds$.
- S^1 acts isometrically on $\mathcal{O}_g$: $A_\cdot \to A_{\cdot+h}$.
- $K = \dot{A}$, $\left|K\right|^2 = \int_{S^1} |\dot{A}_s|^2 ds$.
- Replace DH, BV integral for $p_t(g)$ by $p_t(g) = \int_{\mathcal{O}_g} \exp\left(-\left(E+\omega\right)/t\right) \exp\left(-b^4 d_K K'/2\right)$.
- As $b \to +\infty$, the integral should localize on $(K=0) = (\dot{A}=0)$= geodesics in $\mathcal{O}_g$.

The Gaussian proof in infinite dimensions

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2 ds, |K|^2 = \int_{S^1} |\ddot{g}|^2 ds$.

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2 ds, |K|^2 = \int_{S^1} |\ddot{g}|^2 ds$.
- $E + b^4 |K|^2 /2 = \frac{1}{2}\int_{S^1} \left(|\dot{g}|^2 + \frac{b^4}{2} |\ddot{g}|^2\right) ds \ldots$

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2\, ds, |K|^2 = \int_{S^1} |\ddot{g}|^2\, ds.$
- $E + b^4 |K|^2/2 = \frac{1}{2}\int_{S^1} \left(|\dot{g}|^2 + \frac{b^4}{2}|\ddot{g}|^2\right) ds \ldots$
- $\ldots = \frac{1}{2}\int_{S^1} |b^2\ddot{g} + \dot{g}|^2\, ds.$

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2 \, ds, |K|^2 = \int_{S^1} |\ddot{g}|^2 \, ds.$
- $E + b^4 |K|^2 /2 = \frac{1}{2}\int_{S^1} \left(|\dot{g}|^2 + \frac{b^4}{2} |\ddot{g}|^2\right) ds \ldots$
- $\ldots = \frac{1}{2}\int_{S^1} |b^2\ddot{g} + \dot{g}|^2 \, ds.$
- $\ldots b \to +\infty$ localizes integral on $\ddot{g} = 0$.

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2 ds, |K|^2 = \int_{S^1} |\ddot{g}|^2 ds$.
- $E + b^4 |K|^2 /2 = \frac{1}{2}\int_{S^1} \left(|\dot{g}|^2 + \frac{b^4}{2}|\ddot{g}|^2\right) ds\ldots$
- $\ldots = \frac{1}{2}\int_{S^1} |b^2\ddot{g} + \dot{g}|^2 ds$.
- $\ldots b \to +\infty$ localizes integral on $\ddot{g} = 0$.
- $b^2\ddot{g} + \dot{g} = \dot{w}$ Gaussian.

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2 \, ds, |K|^2 = \int_{S^1} |\ddot{g}|^2 \, ds.$
- $E + b^4 |K|^2 /2 = \frac{1}{2}\int_{S^1} \left(|\dot{g}|^2 + \frac{b^4}{2} |\ddot{g}|^2 \right) ds \ldots$
- $\ldots = \frac{1}{2}\int_{S^1} |b^2\ddot{g} + \dot{g}|^2 \, ds.$
- $\ldots b \to +\infty$ localizes integral on $\ddot{g} = 0$.
- $b^2\ddot{g} + \dot{g} = \dot{w}$ Gaussian.
- If this method works, we will get the required formula!

The Gaussian proof in infinite dimensions

- Recall that $\dot{g} = dgg^{-1} = -A$.
- $E = \frac{1}{2}\int_{S^1} |\dot{g}|^2 \, ds, |K|^2 = \int_{S^1} |\ddot{g}|^2 \, ds.$
- $E + b^4 |K|^2 /2 = \frac{1}{2}\int_{S^1} \left(|\dot{g}|^2 + \frac{b^4}{2} |\ddot{g}|^2 \right) ds \ldots$
- $\ldots = \frac{1}{2}\int_{S^1} |b^2\ddot{g} + \dot{g}|^2 \, ds.$
- $\ldots b \to +\infty$ localizes integral on $\ddot{g} = 0$.
- $b^2\ddot{g} + \dot{g} = \dot{w}$ Gaussian.
- If this method works, we will get the required formula!
- To make it rigorous, we need to introduce counterpart as operators.

The Dirac operator of Kostant

The Dirac operator of Kostant

- B invariant scalar product on $\mathfrak{g}$.
- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G).

The Dirac operator of Kostant

- B invariant scalar product on $\mathfrak{g}$.
- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G).
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.

The Dirac operator of Kostant

- B invariant scalar product on $\mathfrak{g}$.
- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G).
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).

The Dirac operator of Kostant

- B invariant scalar product on $\mathfrak{g}$.
- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G).
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).
- $\widehat{D}^{\mathrm{Ko}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ Dirac operator of Kostant.

The Dirac operator of Kostant

- B invariant scalar product on $\mathfrak{g}$.
- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G).
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).
- $\widehat{D}^{\mathrm{Ko}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ Dirac operator of Kostant.
- $\kappa^{\mathfrak{g}}(U, V, W) = B([U, V], W)$ closed 3-form.

The Dirac operator of Kostant

- B invariant scalar product on $\mathfrak{g}$.
- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G).
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).
- $\widehat{D}^{\mathrm{Ko}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ Dirac operator of Kostant.
- $\kappa^{\mathfrak{g}}(U, V, W) = B([U, V], W)$ closed 3-form.
- $\widehat{D}^{\mathrm{Ko}} = \widehat{c}(e_i^*) e_i + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{g}})$.

Casimir and Kostant

Casimir and Kostant

- $\widehat{D}^{\mathrm{Ko}}$ elliptic first order operator on G.

Casimir and Kostant

- $\widehat{D}^{\mathrm{Ko}}$ elliptic first order operator on G.
- $\widehat{D}^{\mathrm{Ko},2} = \Delta^G - 4\pi^2 \left|\rho\right|^2$.

Casimir and Kostant

- $\widehat{D}^{\mathrm{Ko}}$ elliptic first order operator on G.
- $\widehat{D}^{\mathrm{Ko},2} = \Delta^G - 4\pi^2 \left|\rho\right|^2$.
- Δ^G acts on $C^\infty\left(G, \mathbf{R}\right)$, $\widehat{D}^{\mathrm{Ko}}$ on $C^\infty\left(G, \mathbf{R}\right) \otimes \Lambda^{\cdot}\left(\mathfrak{g}^*\right)$.

Casimir and Kostant

- $\widehat{D}^{\mathrm{Ko}}$ elliptic first order operator on G.
- $\widehat{D}^{\mathrm{Ko},2} = \Delta^G - 4\pi^2 \left|\rho\right|^2$.
- Δ^G acts on $C^\infty\left(G, \mathbf{R}\right)$, $\widehat{D}^{\mathrm{Ko}}$ on $C^\infty\left(G, \mathbf{R}\right) \otimes \Lambda^{\cdot}\left(\mathfrak{g}^*\right)$.
- We will delete $\Lambda^{\cdot}\left(\mathfrak{g}^*\right)$ by tensoring with $S^{\cdot}\left(\mathfrak{g}^*\right)$, and use Bargmann isomorphism.

de Rham, Witten, and Bargmann

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.
- $d^{\mathfrak{g}*} = i_Y$, $[d^{\mathfrak{g}}, i_Y] = L_Y = N^{\mathcal{A}^{\cdot}(\mathfrak{g}^*)}$.

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.
- $d^{\mathfrak{g}^*} = i_Y$, $[d^{\mathfrak{g}}, i_Y] = L_Y = N^{\mathcal{A}^{\cdot}(\mathfrak{g}^*)}$.
- Poincaré lemma and Hodge theory hold.

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.
- $d^{\mathfrak{g}^*} = i_Y$, $[d^{\mathfrak{g}}, i_Y] = L_Y = N^{\mathcal{A}^{\cdot}(\mathfrak{g}^*)}$.
- Poincaré lemma and Hodge theory hold.
- Via Bargmann, $\overline{S}^{\cdot}(\mathfrak{g}^*) \simeq L^2(\mathfrak{g})$, $d^{\mathfrak{g}} + d^{\mathfrak{g}^*}$ maps to $\frac{1}{\sqrt{2}}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}^*} + i_Y\right)\dots$

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.
- $d^{\mathfrak{g}^*} = i_Y$, $[d^{\mathfrak{g}}, i_Y] = L_Y = N^{\mathcal{A}^{\cdot}(\mathfrak{g}^*)}$.
- Poincaré lemma and Hodge theory hold.
- Via Bargmann, $\overline{S}^{\cdot}(\mathfrak{g}^*) \simeq L^2(\mathfrak{g})$, $d^{\mathfrak{g}} + d^{\mathfrak{g}^*}$ maps to $\frac{1}{\sqrt{2}}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}^*} + i_Y\right)\ldots$
- ...and L_Y to $H + N^{\Lambda^{\cdot}(\mathfrak{g}^*)}\ldots$

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.
- $d^{\mathfrak{g}*} = i_Y$, $[d^{\mathfrak{g}}, i_Y] = L_Y = N^{\mathcal{A}^{\cdot}(\mathfrak{g}^*)}$.
- Poincaré lemma and Hodge theory hold.
- Via Bargmann, $\overline{S}^{\cdot}(\mathfrak{g}^*) \simeq L^2(\mathfrak{g})$, $d^{\mathfrak{g}} + d^{\mathfrak{g}*}$ maps to $\frac{1}{\sqrt{2}}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)\ldots$
- $\ldots$ and L_Y to $H + N^{\Lambda^{\cdot}(\mathfrak{g}^*)}\ldots$
- with $H = \frac{1}{2}\left(-\Delta^{\mathfrak{g}} + |Y|^2 - n\right)$ harmonic oscillator.

de Rham, Witten, and Bargmann

- $\mathcal{A}^{\cdot}(\mathfrak{g}^*) = S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ polynomial forms on $\mathfrak{g}$.
- $(\mathcal{A}^{\cdot}(\mathfrak{g}^*), d^{\mathfrak{g}})$ algebraic de Rham complex.
- $d^{\mathfrak{g}*} = i_Y$, $[d^{\mathfrak{g}}, i_Y] = L_Y = N^{\mathcal{A}^{\cdot}(\mathfrak{g}^*)}$.
- Poincaré lemma and Hodge theory hold.
- Via Bargmann, $\overline{S}^{\cdot}(\mathfrak{g}^*) \simeq L^2(\mathfrak{g})$, $d^{\mathfrak{g}} + d^{\mathfrak{g}*}$ maps to $\frac{1}{\sqrt{2}}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)\ldots$
- $\ldots$ and L_Y to $H + N^{\Lambda^{\cdot}(\mathfrak{g}^*)}\ldots$
- with $H = \frac{1}{2}\left(-\Delta^{\mathfrak{g}} + |Y|^2 - n\right)$ harmonic oscillator.
- These operators act on $C^{\infty}(\mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.

A deformed Dirac operator

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{\sqrt{2}b}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)$.

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{\sqrt{2}b}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)$.
- As $b \to 0$, $\mathfrak{D}_b$ deforms 0 operator acting on $C^{\infty}(G, \mathbf{R})$.

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{\sqrt{2}b}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)$.
- As $b \to 0$, $\mathfrak{D}_b$ deforms 0 operator acting on $C^{\infty}(G, \mathbf{R})$.
- $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{\sqrt{2}b}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)$.
- As $b \to 0$, $\mathfrak{D}_b$ deforms 0 operator acting on $C^{\infty}(G, \mathbf{R})$.
- $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- $\mathcal{L}_b = \frac{1}{2b^2}\left(-\Delta^{\mathfrak{g}} + |Y|^2 - n\right) + \frac{1}{b}\left(\nabla_Y + \widehat{c}(\mathrm{ad}(Y))\right)$.

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{\sqrt{2}b}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)$.
- As $b \to 0$, $\mathfrak{D}_b$ deforms 0 operator acting on $C^{\infty}(G, \mathbf{R})$.
- $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- $\mathcal{L}_b = \frac{1}{2b^2}\left(-\Delta^{\mathfrak{g}} + |Y|^2 - n\right) + \frac{1}{b}\left(\nabla_Y + \widehat{c}(\mathrm{ad}(Y))\right)$.
- $\mathcal{L}_b$ hypoelliptic Laplacian.

A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{\sqrt{2}b}\left(d^{\mathfrak{g}} + Y \wedge + d^{\mathfrak{g}*} + i_Y\right)$.
- As $b \to 0$, $\mathfrak{D}_b$ deforms 0 operator acting on $C^{\infty}(G, \mathbf{R})$.
- $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- $\mathcal{L}_b = \frac{1}{2b^2}\left(-\Delta^{\mathfrak{g}} + |Y|^2 - n\right) + \frac{1}{b}\left(\nabla_Y + \widehat{c}\left(\mathrm{ad}\left(Y\right)\right)\right)$.
- $\mathcal{L}_b$ hypoelliptic Laplacian.
- As $b \to 0$, by collapsing, $\mathcal{L}_b$ deforms $\frac{1}{2}\left(-\Delta^G + c\right)$.

The invariance of the trace under deformation

The invariance of the trace under deformation

- $\int_G p_t\left(g_1 g g_2^{-1} g^{-1}\right) dg = \mathrm{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\Delta^G/2\right)\right].$

The invariance of the trace under deformation

- $\int_G p_t\left(g_1 g g_2^{-1} g^{-1}\right) dg = \operatorname{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\Delta^G/2\right)\right].$

Theorem (B08)

For $t > 0, g_1, g_2 \in G$,

$$\operatorname{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\left(\Delta^G - c\right)/2\right)\right] = \operatorname{Tr_s}\left[L_{g_1} R_{g_2}^{-1} \exp\left(-t\mathcal{L}_b\right)\right].$$

The invariance of the trace under deformation

- $\int_G p_t\left(g_1 g g_2^{-1} g^{-1}\right) dg = \mathrm{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\Delta^G/2\right)\right].$

Theorem (B08)

For $t > 0, g_1, g_2 \in G$,

$$\mathrm{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\left(\Delta^G - c\right)/2\right)\right] = \mathrm{Tr_s}\left[L_{g_1} R_{g_2}^{-1} \exp\left(-t\mathcal{L}_b\right)\right].$$

Independence of b like in index theorem, and convergence as $b \to 0$.

The invariance of the trace under deformation

- $\int_G p_t\left(g_1 g g_2^{-1} g^{-1}\right) dg = \mathrm{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\Delta^G/2\right)\right]$.

Theorem (B08)

For $t > 0, g_1, g_2 \in G$,

$$\mathrm{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\left(\Delta^G - c\right)/2\right)\right] = \mathrm{Tr_s}\left[L_{g_1} R_{g_2}^{-1} \exp\left(-t\mathcal{L}_b\right)\right].$$

Independence of b like in index theorem, and convergence as $b \to 0$.

Remark

Hamiltonian counterpart to Lagrangian deformation for DH, BV formulas.

The limit as $b \to +\infty$

The limit as $b \to +\infty$

- $\mathcal{L}_b \simeq \frac{1}{2}\left(-\frac{\Delta^{\mathfrak{g}}}{b^4} + \left|Y\right|^2 - \frac{n}{b^2}\right) + \nabla_Y + \widehat{c}\left(\mathrm{ad}\left(Y\right)\right).$

The limit as $b \to +\infty$

- $\mathcal{L}_b \simeq \frac{1}{2}\left(-\frac{\Delta^{\mathfrak{g}}}{b^4} + |Y|^2 - \frac{n}{b^2}\right) + \nabla_Y + \widehat{c}(\mathrm{ad}\,(Y))$.
- As $b \to +\infty$, geodesic flow dominates.

The limit as $b \to +\infty$

- $\mathcal{L}_b \simeq \frac{1}{2}\left(-\frac{\Delta^{\mathfrak{g}}}{b^4} + |Y|^2 - \frac{n}{b^2}\right) + \nabla_Y + \widehat{c}(\mathrm{ad}(Y))$.
- As $b \to +\infty$, geodesic flow dominates.
- Forces localization of trace on geodesics.

The limit as $b \to +\infty$

- $\mathcal{L}_b \simeq \frac{1}{2}\left(-\frac{\Delta^{\mathfrak{g}}}{b^4} + |Y|^2 - \frac{n}{b^2}\right) + \nabla_Y + \widehat{c}(\mathrm{ad}(Y))$.
- As $b \to +\infty$, geodesic flow dominates.
- Forces localization of trace on geodesics.
- We get the required formulas for the above trace in terms of the coroot lattice.

Applications

Applications

- If G is instead a reductive group, we get geometric formulas for semi-simple orbital integrals...

Applications

- If G is instead a reductive group, we get geometric formulas for semi-simple orbital integrals…
- …by following a similar strategy.

Applications

- If G is instead a reductive group, we get geometric formulas for semi-simple orbital integrals...
- ...by following a similar strategy.
- The geometric and analytic difficulties are bigger!

The Dirac operator on $\widetilde{LG}$

The Dirac operator on $\widetilde{LG}$

- Like in index theory, we have been able to deal with DH,BV infinite dimensional analogues.

The Dirac operator on $\widetilde{LG}$

- Like in index theory, we have been able to deal with DH,BV infinite dimensional analogues.
- What is needed now is infinite dimensional index theory!

The Dirac operator on $\widetilde{LG}$

- Like in index theory, we have been able to deal with DH,BV infinite dimensional analogues.
- What is needed now is infinite dimensional index theory!
- More precisely, we would need a good Dirac operator on $\mathcal{O}_g \ldots$

The Dirac operator on $\widetilde{LG}$

- Like in index theory, we have been able to deal with DH,BV infinite dimensional analogues.
- What is needed now is infinite dimensional index theory!
- More precisely, we would need a good Dirac operator on $\mathcal{O}_g$...
- ...to obtain an index theoretic construction of the representations of $\widetilde{LG}$.

The Dirac operator on $\widetilde{LG}$

- Like in index theory, we have been able to deal with DH,BV infinite dimensional analogues.
- What is needed now is infinite dimensional index theory!
- More precisely, we would need a good Dirac operator on $\mathcal{O}_g \ldots$
- …to obtain an index theoretic construction of the representations of $\widetilde{LG}$.
- Connections with Verlinde formulas…

- I. B. Frenkel, *Orbital theory for affine Lie algebras*, Invent. Math. **77** (1984), no. 2, 301–352. MR MR752823 (86d:17014)
- M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, Astérisque (1985), no. 131, 43–59, Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). MR 87h:58206
- J.-M. Bismut, *The hypoelliptic Laplacian on a compact Lie group*, J. Funct. Anal. **255** (2008), no. 9, 2190–2232. MR MR2473254

______, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080