Quantum Speed Limit and Relative Categorical Energy

Sheng-Fu Chiu

NCTS

Cologne, July 24 - 28, 2023

Outline

1. Uncertainty Principle and Quantum Speed Limit
2. Symplectic Displacement and Quantum Speed Limit
3. Modeling with Derived Categories
4. Categorical Displacement Energy

Heisenberg Uncertainty Principle

- $\psi = \frac{\psi}{|\psi|} \in$ Hilbert space $\mathcal{H}$ pure quantum state
- $O \in \{$hermitian operators on $\mathcal{H}\}$ quantum observable
- $\langle O \rangle := \langle \psi | O | \psi \rangle$ expectation value
- $\triangle O := \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$ standard deviation (uncertainty)

$\mathcal{H} = L^2(\mathbb{R}^n)$ suitable function space $\Rightarrow \triangle O_1 \triangle O_2 \geq \frac{1}{2}\langle [O_1, O_2] \rangle$
When $O_1 = \hat{q}_1$, $O_2 = \hat{p}_1 = \frac{\hbar}{i}\frac{\partial}{\partial q_1}$ (O_1 and O_2 are conjugated)

Heisenberg Uncertainty Principle:

$$\triangle \hat{q}_1 \triangle \hat{p}_1 \geq \frac{\hbar}{2} \quad \leftarrow \text{independent of } \psi$$

deduced from noncommutativity relation $[\hat{q}_1, \hat{p}_1] = i\hbar$.
Noncommutativity=**Obstruction for simultaneous measurement**.

Conservation Law

Noether Theorem: for conjugated physical quantities O_1, O_2, if the physical law does not depend on O_1, then the quantity O_2 is conserved under system evolution.

- Translation-Symmetry $\Rightarrow$ Conservation of Linear Momentum
- Rotation-Symmetry $\Rightarrow$ Conservation of Angular Momentum
- Phase-Symmetry $\Rightarrow$ Conservation of Charge
- Time-Symmetry $\Rightarrow$ Conservation of Total Energy

Heisenberg Uncertainty Principle is a statistical variant of Noether Conservation Law. So it is natural to ask

What is **Energy-Time Uncertainty** ?

Mysterious Energy-Time Uncertainty

When it comes to Energy-Time Uncertainty, the notion of simultaneous measurement becomes troublesome because:

1. Time is not a quantum observable
2. All observables can be measured with arbitrary accuracy in arbitrary short time

In the famous *Bohr-Einstein Debates*, Einstein demonstrated that fixed small $\triangle t$, we could measure E precisely using $E = mc^2$. But Bohr argued that the physical measurement of the mass m relies on a mechanical design against the gravity of Earth. Therefore, by General Relativity, such displacement in the gravitational field yields an intrinsic uncertainty of time duration it experiences. (Still problematic, of course)

Quantum Speed Limit (Mandelstam-Tamm 1945)

The mysterious Energy-Time Uncertainty is not about simultaneous measurement nor Relativity, but rather **speed of quantum evolution!**

Let H be the quantum Hamiltonian operator that governs the quantum evolution of the state $\psi(t)$. Consider the **orthogonal time** τ_{orth} at which the evolution driven by H **dislocates** the initial state, that is $\psi(\tau) \perp \psi(0)$. They showed that

$$\tau_{orth} \geq \frac{\pi\hbar}{2}\frac{1}{\triangle H} := \tau_{QSL}. \leftarrow \text{independent of } \psi$$

QSL = Minimal Orthogonal Time

Quantum Speed Limit (Margolus-Levitin 1998)

$\triangle H$ may diverge. They showed that for $\langle H \rangle > 0$ with zero ground energy,

$$\tau_{orth} \geq \frac{\pi\hbar}{2}\max\{\frac{1}{\triangle H}, \frac{1}{\langle H \rangle}\} := \tau_{QSL}. \leftarrow \text{independent of } \psi$$

1. Without referring to noncommutativity relation
2. τ_{QSL} sets an universal bound of **minimal time** for the system to evolves from one state to an orthogonal state with given **energy**
3. Being **orthogonal** = being distinguishable
4. τ_{QSL} sets an intrinsic scale for quantum computational capability
5. Both MT and ML limits have been tested for single atom in an optical trap (*Sci. Adv., 2021*)

Quantum-Classical Mechanics (after 2018)

The formation

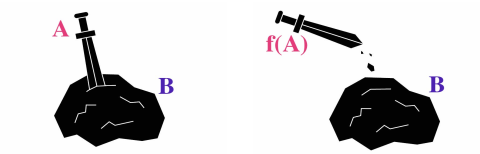
$$\tau_{orth} \geq \tau_{QSL} := \frac{\pi\hbar}{2}\max\{\frac{1}{\triangle H}, \frac{1}{\langle H\rangle}\}$$

leads to $\lim_{\hbar\to 0}\tau_{QSL} = 0$. But it does not mean τ_{orth} vanishes as $\hbar \to 0$!
Two surprising papers on *Phys.Rev.Lett.*:

1. *QSL Is Not Quantum* (Okuyama-Ohzeki)
2. *QSL Across Quantum-Classical Transition* (Shanahan-Chenu-Margolus-del Campo)

As many-particle effects they obtained nontrivial **Speed Limits** for Liouville equations and Wigner representations respectively. Both speed limits can be derived from **dynamical properties of Hilbert space under unitary evolution**.

Displacement Problem in Symplectic Topology



Given $A, B \subset (M, \omega)$. We say A is **displaceable** from B if $\exists f = (f_s)|_{s=1} \in Ham(M)$ such that

$$f(A) \cap B = \emptyset.$$

For example $M = S^2$, $A = B = S^1$

1. if A is a big circle then A is **not displaceable** from itself
2. if A is a small circle then A is **displaceable** from itself

Displacement Energy

More quantitatively, the **Hofer displacement energy** is defined by

$$e(A, B) := \inf\{\|F\| \,:\, F \rightsquigarrow f, f(A) \cap B = \emptyset\},$$

where the norm of Hamiltonian function $F : M \times I \to \mathbb{R}$ is defined by

$$\|F\| := \int_0^1 (\max_M F_s - \min_M F_s) ds$$

which is L^1 in time and L^∞ in phase space.

When A is not displaceable from B we denote by $e(A, B) = \infty$.

Quantitative Displacement in Symplectic Topology

Some facts for closed submanifolds $A, B \subset M$:

1. if $\dim A + \dim B < \dim M$ then $e(A, B) = 0$
2. if $\dim(A) = \frac{1}{2} \dim M$ but A is not Lagrangian and there is no topological obstruction, then $e(A, A) = 0$
3. if A, B are Lagrangians and $HF^\bullet(A, B) \neq 0$, then $e(A, B) = \infty$

On the other hand, if $B \subset M$ has non-empty interior then

$$e(B, B) > 0.$$

QSL and Self-displacement (Charles-Polterovich 2018)

Charles and Polterovich discover (*Ann. Henri Poincaré*) a link between QSL and self-displacement energy $e(B, B)$ where they represent

- B by the support $supp(\theta)$ of a **pre-quantized state** θ
- Hamiltonian diffeomorphism f by a **pre-quantized unitary evolution**
- Displacement of B by **orthogonality (dislocation) of quantum states**,

and ask for the **least energy** for any flow to displace $B = supp(\theta)$ in a **unit of time**.

Dually, QSL asks for the minimal orthogonal time (in terms of Hamiltonian) universal for any initial quantum state.

Coherent Berezin–Toeplitz Quantization

(M, ω) quantizable closed Kähler manifold. For $1/\hbar \in \mathbb{N}$, there exists Hilbert space $\mathcal{H}_\hbar$ and quantization maps $Q_\hbar : \mathrm{Prob}(M) \to \mathcal{S}(\mathcal{H}_\hbar)$, $T_\hbar : C^\infty(M) \to \mathcal{L}(\mathcal{H}_\hbar)$.

Table	**Classical Dynamics**	**Quantum Dynamics**
State	$supp(\theta) = B \subset M$	$Q_\hbar(\theta) \in \mathcal{S}(\mathcal{H}_\hbar)$
Evolution	Hamiltonian diffeomorphism $F \rightsquigarrow f : M \to M$	Unitary operator $T_\hbar(F) \rightsquigarrow U_\hbar(F) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$
Energy	$\int_0^1 \lVert f_s \rVert ds$	$\int_0^1 \lVert T_{\hbar,s}(F) \rVert_{op} ds$.

Charles and Polterovich establish connections between the followings:

1. f displaces $supp(\theta)$.
2. $T_\hbar(F)$ $\hbar$-asymptotically dislocates $Q_\hbar(\theta)$.
3. Quantum energy $\int_0^1 \lVert T_{\hbar,s}(F) \rVert_{op} ds \geq e(B, B)$ $\hbar$-asymptotically.

Quantitative Displacement of Mixed Type

When A is Lagrangian and B is open,

Question

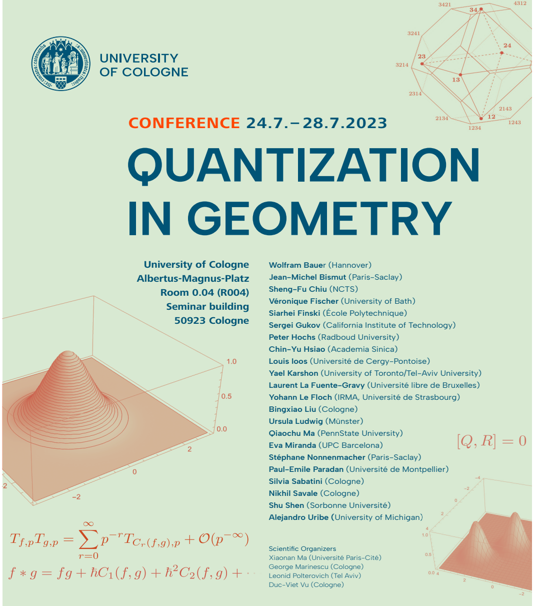
Given $A \cap B \neq \emptyset$. What can we say about $e(A, B)$?

Oh (2018) proves the strictly positivity of $e(A, B)$ for more general (M, ω) using techniques of pseudo-holomorphic disks. Here we would like to focus on the case of cotangent bundles, using a **QSL-like** argument to characterize the energy $e(A, B)$ where we represent

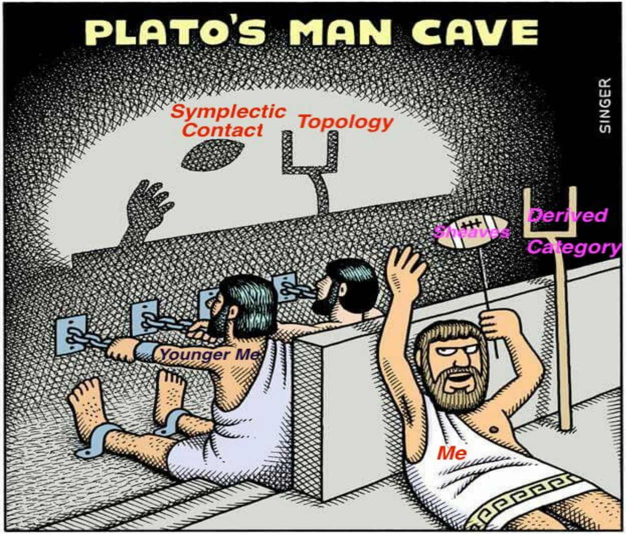
- Lagrangian A by a **quantum-like state**
- Open B by a **collection of quantum-like states**
- Hamiltonian diffeomorphism f by a **unitary-like system evolution**
- Displacement by **orthogonality of states**.

We ask for the **least energy** to dislocate a given state from a given collection of states in a **unit of time**.

What people think we do



What we actually do



(Cartoon provided by Andy Singer)

Modeling with Derived Categories

Fix a ground ring $\mathbb{k}$.
$D(Q)$ = derived category of sheaves of $\mathbb{k}$-modules over manifold Q.
Work in a slightly refined dg-triangulated category $\mathcal{D}$ (described later).

1. $\mathcal{D}$ is our derived space of states
2. $Rhom : \mathcal{D}^{op} \times \mathcal{D} \to D(\mathbb{k}\text{-mod})$ is our derived inner product which measures mutual overlapping of states
3. Instead of orthogonality we have left/right semiorthogonality
4. A natural distance d on $\mathcal{D}$
5. Any $f \in Ham(T^*Q)$ induces an autoequivalence $\mathcal{S}(f) : \mathcal{D} \to \mathcal{D}$.

Microsupport à la Kashiwara-Schapira

	Quantum Dynamics	Classical Dynamics	Sheaf Theory
State	$\psi \in \mathcal{H}$	Lagrangian $L \subset T^*Q$	$\mathcal{G} \in \mathcal{D}(Q \times \mathbb{R})$
Subsystem	Subspace $\mathcal{V} \subset \mathcal{H}$	Open set $B \subset T^*Q$	Subcategory $\mathcal{D}_B(Q \times \mathbb{R}) \subset \mathcal{D}(Q \times \mathbb{R})$
Evolution	Unitary operator $U : \mathcal{H} \to \mathcal{H}$	Hamiltonian diffeomorphism $f : T^*Q \to T^*Q$	Convolution functor $\bullet\mathcal{S} : \mathcal{D}(Q \times \mathbb{R}) \to \mathcal{D}(Q \times \mathbb{R})$
Fidelity	L^2-inner product	Set-theoretic intersection	Derived homomorphisms *Rhom*

A bridge between algebra and geometry $SS : \{\text{sheaves}\} \to \{\text{sets}\}$.
For $\mathcal{G} \in D(Q)$, define its **microsupport** $SS(\mathcal{G}) \subset T^*Q$ by the closure of those (q_0, p_0) such that $\exists \phi : Q \xrightarrow{C^1} \mathbb{R}$ and $d\phi(q_0) = p_0$ satisfying

$$(R\Gamma_{\{q \in Q | \phi(q) \geq \phi(q_0)\}}\mathcal{G})_{q_0} \not\cong 0.$$

$SS(\mathcal{G})$ is the closed conic subset consists of **singular codirections**, i.e., codirections along which the derived sections of $\mathcal{G}$ cease to propagate. $SS(\mathcal{G})$ is cone-coisotropic in general.

Subcategory as collection of states

For geometric reason instead of $D(Q)$ we choose to work in a more "faithful" category $\mathcal{D}(Q \times \mathbb{R})$. Let

$$\rho : T^*(Q \times \mathbb{R}) = \{(q, p, z, \zeta)\} \to \{(q, \frac{p}{\zeta})\} = T^*Q$$

and

$$D_{\zeta \leq 0} = \{\mathcal{G} \in D(Q \times \mathbb{R}) | SS(\mathcal{G}) \subset \{\zeta \leq 0\}\}.$$

Definition (Tamarkin Category)

- $\mathcal{D} := D_{\zeta \leq 0}(Q \times \mathbb{R})^{\text{left}\perp}$ w.r.t. *Rhom* in $D(Q \times \mathbb{R})$
- $\mathcal{D}_A := \{\mathcal{G} \in \mathcal{D} | SS(\mathcal{G}) \subset \rho^{-1}(A)\}$, $\forall A \overset{\text{cls}}{\subset} T^*Q$
- $\mathcal{D}_B := D_{T^*Q \setminus B}{}^{\text{left}\perp}$ w.r.t *Rhom* in $\mathcal{D}$, $\forall B \overset{\text{open}}{\subset} T^*Q$.

Interleaving Distance on $\mathcal{D}$

Let $T_a : z \mapsto z + a$ acting on $D(Q \times \mathbb{R})$. A remarkable feature of $\mathcal{D}$ is we have for $a \geq 0$, a natural transformation between endofunctors

$$\tau_a : Id \Rightarrow T_a.$$

Let $\mathcal{F}, \mathcal{G} \in \mathcal{D}(Q \times \mathbb{R})$ and $a, b \geq 0$. We say the pair $(\mathcal{F}, \mathcal{G})$ is (a, b)-**interleaved** if there exists morphisms $\mathcal{F} \underset{\delta}{\overset{\alpha}{\rightrightarrows}} T_a\mathcal{G}$ and $\mathcal{G} \underset{\gamma}{\overset{\beta}{\rightrightarrows}} T_b\mathcal{F}$ satisfying

$$\begin{cases} [\mathcal{F} \xrightarrow{\alpha} T_a\mathcal{G} \xrightarrow{T_a\beta} T_{a+b}\mathcal{F}] = \tau_{a+b}(\mathcal{F}), \\ [\mathcal{G} \xrightarrow{\gamma} T_b\mathcal{F} \xrightarrow{T_b\delta} T_{a+b}\mathcal{G}] = \tau_{a+b}(\mathcal{G}). \end{cases}$$

The **interleaving distance** is defined by

$$d(\mathcal{F}, \mathcal{G}) := \inf\{a + b \,|\, (\mathcal{F}, \mathcal{G}) \text{ is } (a, b)\text{-interleaved}\}.$$

Glossary

	Quantum Dynamics	**Classical Dynamics**	**Sheaf Theory**
State	$\psi \in \mathcal{H}$	Lagrangian $L \subset T^*Q$	$\mathcal{G} \in \mathcal{D}(Q \times \mathbb{R})$
Subsystem	Subspace $\mathcal{V} \subset \mathcal{H}$	Open set $B \subset T^*Q$	Subcategory $\mathcal{D}_B(Q \times \mathbb{R}) \subset \mathcal{D}(Q \times \mathbb{R})$
Evolution	Unitary operator $U : \mathcal{H} \to \mathcal{H}$	Hamiltonian diffeomorphism $f : T^*Q \to T^*Q$	Convolution functor $\bullet\mathcal{S} : \mathcal{D}(Q \times \mathbb{R}) \to \mathcal{D}(Q \times \mathbb{R})$
Fidelity	L^2-inner product	Set-theoretic intersection	Derived homomorphisms *Rhom*
Displacement from given subsystem	$U(\psi) \in \mathcal{V}^\perp$	$f(A) \cap B = \emptyset$	$\mathcal{F} \bullet \mathcal{S} \in \mathcal{D}_B{}^{\text{left}\perp}$
Energy of given evolution	Operator norm $\int_0^1 \|U_s\|_{\text{op}} ds$	Hofer norm $\int_0^1 \|f_s\|_{\text{Hofer}} ds$	Interleaving distance $(-) \mapsto d((-) \bullet \mathcal{S}, -)$

Microlocal Projector

The advantage of modelling in microlocal sheaf theory is the following existence theorem of microlocal projector:

Theorem (C.)

Let B be a bounded open subset of T^*Q, then in $\mathcal{D}(Q \times Q \times \mathbb{R})$ there exists an exact triangle ($\mathcal{P}_B \to \mathbb{K}_\Delta \to \mathcal{Q}_B \xrightarrow{+1}$) such that the convolution with the above triangle gives rise to the semiorthogonal decomposition with respect to the triple of subset categories $(\mathcal{D}_B(Q \times \mathbb{R}), \mathcal{D}(Q \times \mathbb{R}), \mathcal{D}_{T^*Q \setminus B}(Q \times \mathbb{R}))$.

Fantastic Beasts and Where to Find Them

$\mathcal{P}_B$ admits a **right-adjoint** functor $\mathcal{E}_B$:

$$Rhom(\mathcal{F} \bullet \mathcal{P}_B, \mathcal{G}) \cong Rhom(\mathcal{F}, \mathcal{E}_B(\mathcal{G})).$$

$$\begin{array}{ccccc} & \overset{\bullet\mathcal{P}_B}{\swarrow} & \mathcal{D} & \overset{\mathcal{E}_B}{\searrow} & \\ & & \cup & & \\ {\mathcal{D}_{T^*Q\setminus B}}^{\text{left}\perp} & = & \mathcal{D}_B & = & {\mathcal{D}_{T^*Q\setminus B}}^{\text{right}\perp} \end{array}$$

The functor $\mathcal{E}_B$ is Lipshitz with respect to the interleaving distance:

$$d(\mathcal{E}_B(\mathcal{F}), \mathcal{E}_B(\mathcal{G})) \leq d(\mathcal{F}, \mathcal{G}).$$

Categorical Energy

Definition (Categorical Energy relative to B)

For $\mathcal{G} \in \mathcal{D}(Q \times \mathbb{R})$ we define $e_B(\mathcal{G}) := d(0, \mathcal{E}_B(\mathcal{G}))$.

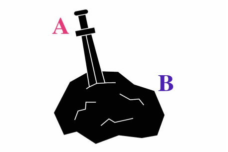
For suitable choice of $\mathcal{G} \in \mathcal{D}_A$ a nontrivial lower bound estimate of $e_B(\mathcal{G})$ is available.

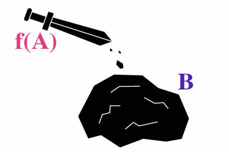
In T^*R^n, let $B = B(r) = \{q^2 + p^2 < r^2\}$ be a standand open ball and let $\mathcal{G} = \Bbbk_{\mathbb{R}^n \times \mathbb{R}_{\geq 0}} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ be the sheaf quantization of the zero section $\mathbb{R}^n \times \{p = 0\}$. Our knowledge of $\mathcal{P}_B$ and $\mathcal{E}_B$ enables us to compute:

Proposition (Capacity-like Property)

For such $\mathcal{G}$ and B, $e_B(\mathcal{G}) \geq \frac{1}{2}\pi r^2$.

Energy Comparison





Recall the definition of Hofer displacement energy

$$e(A,B) := \inf_{F \rightsquigarrow f} \{ \int_0^1 (\max_M F_s - \min_M F_s) ds \, : \, f(A) \cap B = \emptyset \}.$$

Theorem (Comparison with Hofer Energy)

Given A closed and B open. Then for any $\mathcal{G} \in \mathcal{D}_A(Q \times \mathbb{R})$ one has

$$e(A,B) \geq e_B(\mathcal{G}).$$

Energy Comparison sketch of proof

Theorem (Comparison with Hofer Energy)

Given A closed and B open. Then for any $\mathcal{G} \in \mathcal{D}_A(Q \times \mathbb{R})$ one has

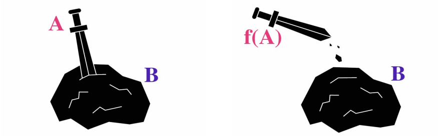
$$e(A, B) \geq e_B(\mathcal{G}).$$

Suppose $f(A) \cap B = \emptyset$ for some $F \rightsquigarrow f \in Ham(T^*Q)$.
By $\mathcal{D}_{f(A)} = \mathcal{D}_A \bullet \mathcal{S}(f)$ one has $\mathcal{E}_B(\mathcal{G} \bullet \mathcal{S}(f)) = 0$.
Therefore

$$e_B(\mathcal{G}) = d(0, \mathcal{E}_B(\mathcal{G})) = d(\mathcal{E}_B(\mathcal{G} \bullet \mathcal{S}(f)), \mathcal{E}_B(\mathcal{G})) \leq d(\mathcal{G} \bullet \mathcal{S}(f), \mathcal{G}) \leq \|F\|.$$

The last inequality is Asano-Ike's inequality (*J. Symplectic Geom., 2020*).

Quantitative Displacement of Mixed Type



Theorem (Relative Energy-Capacity Inequality)

Given A smooth manifold and $B = \jmath(B(r))$ symplectically embedded ball of T^*A relative to A (that is $\jmath^{-1}(A) = \mathbb{R}^n \cap B(r)$). Then $e(A, B) \geq \frac{1}{2}\pi r^2$.

We expect future applications of e_B to the study of Viterbo's γ-distance and Guillermou-Viterbo's γ-coisotropic sets.

Quantum Footprints à la Polterovich

Some examples of **Quantum Footprints** in symplectic geometry and topology:

1. **Uncertainty Principle** $\rightsquigarrow$ Nonsqueezing of Symplectic Balls
2. **Quantized Phase-Energy Levels** $\rightsquigarrow$ Nonsqueezing of Contact Balls
3. **Quantum Unsharpness** $\rightsquigarrow$ Rigidity of Partition of Unity
4. **Quantum Speed Limit** $\rightsquigarrow$ Symplectic Displacement Energy

Thank You !