Submultiplicative norms on section rings

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Quantization in Geometry
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Setting for the talk

X complex projective manifold, $\dim X = n$; L ample line bundle

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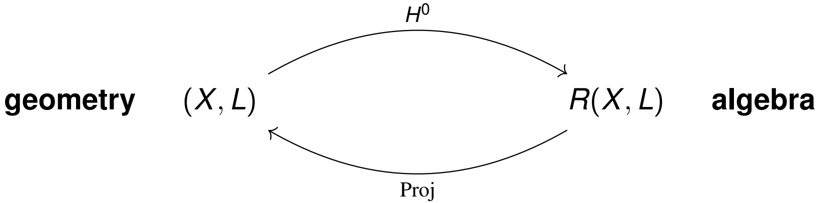

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$$\textbf{geometry} \quad (X, L) \underset{\mathrm{Proj}}{\overset{H^0}{\rightleftarrows}} R(X, L) \quad \textbf{algebra}$$

Today's goal : Understand better the correspondence

metrics $\leftrightsquigarrow$ **norms**

Metrics on $L \to$ norms on $R(X, L)$

Hermitian metric h^L on L

$$\rightsquigarrow$$

L^∞-norms $\mathrm{Ban}_k^\infty(h^L) := \|\cdot\|_{L^\infty(X)}$ on $H^0(X, L^k)$, $k \in \mathbb{N}^*$

$$\|f\|_{L^\infty(X)} := \sup_{x \in X} |f(x)|_{(h^L)^k}, \quad f \in H^0(X, L^k)$$

"Ban" stands for Banach

Submultiplicative norms

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For any submultiplicative graded norm N on $R(X, L)$ [satisfying some hypotheses] there is a unique* metric h^L on L such that

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Plan for the rest of the talk

A what is $\sim$? how to construct h^L ? hypotheses ?
B examples. motivations and applications.
C proof.

Equivalence of norms

Sequences of norms N_k, N'_k on $H^0(X, L^k)$, $k \in \mathbb{N}$, are *equivalent* ($\sim$) if $\forall \epsilon > 0$, $\exists k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$

$$\exp(-\epsilon k) \cdot N_k \leq N'_k \leq \exp(\epsilon k) \cdot N_k$$

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Lemma : $\mathrm{Ban}^\infty(h_0^L) \nsim \mathrm{Ban}^\infty(h_1^L)$ for $h_0^L \neq h_1^L$ continuous psh

h^L is **psh** if locally $h^L = e^{-\phi}$, where ϕ is plurisubharmonic (psh),

h^L is **positive** if it is smooth and ϕ is strictly plurisubharmonic.

Norm N_k on $H^0(X, L^k) \rightarrow$ Fubini-Study metric $FS(N_k)$ on L^k

From norms to metrics

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Defn : let $l \in L^k_x$, $x \in X$, we define $|l|_{FS(N_k)} := \inf_{\substack{s \in H^0(X,L^k) \\ s(x)=l}} \|s\|_k$.

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Fekete's lemma : For $N = \sum N_k$ submultiplicative, as $k \to \infty$, $FS(N_k)^{\frac{1}{k}}$ converge. Let $FS(N) := \lim_{k \to \infty} FS(N_k)^{\frac{1}{k}}$

Submultiplicative norms and Fubini-Study metrics

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Attention ! $FS(N)$ is only upper-semicontinuous ; probably zero

Main theorem of the talk

Theorem (-, 2022)

For any submultiplicative graded norm N on $R(X, L)$ such that $FS(N)$ is continuous and non-zero

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b) Theorem fails with no assumption on $FS(N)$. But it can be remedied by a weaker equivalence relation $\sim$.

Examples

A norm on homogeneous polynomials

V finitely dim. $\mathbb{C}$-vector space, $X = \mathbb{P}(V^*)$, $L = \mathscr{O}(1)$.
$R(X, L)$ = $\mathrm{Sym}(V)$ = space of polynomials on V^*

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Our theorem gives the opposite (nontrivial !) inequality :

$\forall \epsilon > 0$, $\exists k_0$, so that $\forall k \geq k_0$, $\| \cdot \|_{L^\infty(FS(N))} \geq \exp(-\epsilon k) \cdot \| \cdot \|_k^\pi$

Are there natural examples of submultiplicative norms?

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1. $L^{\frac{1}{k}}$-(pseudo)norms : $\|f\|_k := (\int_{x \in X} |f(x)|_{h^L}^{\frac{1}{k}} \cdot dv)^k$
2. Mahler (pseudo)norms : $\|f\|_k := \exp(\int_{x \in X} \log |f(x)|_{h^L} \cdot dv)$
3. Complex interpolation between submultiplicative norms
4. Projective tensor norms on symmetric algebras
5. Quotients of submultiplicative norms
6. Interpolations of submultiplicative filtrations

4. Projective tensor norms on symmetric algebra

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Injective tensor norm $N_1 \otimes_\epsilon N_2 = \|\cdot\|_{\otimes_\epsilon}$. For $f \in V_1 \otimes V_2$

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If we now apply our theorem for $\mathrm{Sym}^\pi N_V$, it would give : $\forall \epsilon > 0$, $\exists k_0$, so that $\forall k \geq k_0$, $P \in \mathrm{Sym}^k V$, $\|P\|_k^\epsilon \geq \exp(-\epsilon k) \cdot \|P\|_k^\pi$

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For any (V, N_V) denote the projective tensor norm by $\mathrm{Sym}^\pi N_V$ on $\mathrm{Sym}\, V = R(\mathbb{P}(V^*), \mathscr{O}(1))$

Observation :$\mathrm{Sym}^\pi N_V$ is submultiplicative. Also, $\|\cdot\|_k^\epsilon \leq \|\cdot\|_k^\pi$

If we now apply our theorem for $\mathrm{Sym}^\pi N_V$, it would give : $\forall \epsilon > 0$, $\exists k_0$, so that $\forall k \geq k_0$, $P \in \mathrm{Sym}^k V$, $\|P\|_k^\epsilon \geq \exp(-\epsilon k) \cdot \|P\|_k^\pi$

Surprising ! In full tensor algebra, the analogue is false for any (V, N_V), $\dim V > 1$, by a result of Pisier, 1980

5. Quotient norms and extension theorem

$Y \subset X$ complex submanifold, the restriction operator

$$\mathrm{Res} : H^0(X, L^k) \twoheadrightarrow H^0(Y, L^k)$$

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$\|f\|_{[N]} := \inf \{ \|g\|_N; \quad g \in R(X, L), \mathrm{Res}(g) = f \}, \quad f \in R(Y, L).$

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If we now take $N = \mathrm{Ban}^\infty(h^L)$, our theorem for $[N]$ gives us : $\forall \epsilon > 0$, $\exists k_0$, so that $\forall k \geq k_0$, $f \in H^0(Y, L^k)$ there is a holomorphic extension $\tilde{f} \in H^0(X, L^k)$ of f, such that

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This is a semiclassical Ohsawa-Takegoshi extension theorem.
Established by S.-W. Zhang 1995 and Bost 2003. In (-, 2021), author refined it by replacing $\exp(\epsilon k)$ by $1 + \frac{C}{k}$.

Filtration on a finitely dimensional vector space V is a map
$\mathcal{F} : \mathbb{Z} \to \{$ vector subspaces of $V\}$

decreasing, i.e. $\mathcal{F}^{n+1}V \subset \mathcal{F}^{n}V$
separating, i.e. $\mathcal{F}^{\infty}V = \{0\}$
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6. Introduction to submultiplicative filtrations

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Motivation : study of submultiplicative filtrations is related to K-stability, itself related with constant scalar curvature metrics

6. Interpolations of submultiplicative filtrations

Let N be a submultiplicative norm on $R(X, L)$ and $\mathcal{F}$ is a submultiplicative filtration. Define ray of norms $N_t = \|\cdot\|_t$, $t \geq 0$

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Our theorem (+ a lot of work, see (-, 2023)), applied for N_t, proves a conjecture of Darvas-Lu 2019 and K. Zhang 2021.

Roughly, conjecture says that :

Geometry at infinity on the space of Kähler potentials
is related with
Asymptotic study of submultiplicative filtrations

Isometry and quantization

Spaces of metrics and norms

$\mathcal{H}^L$ the space of continuous psh metrics on L

$\mathcal{H}_k$ the space of norms on $H^0(X, L^k)$

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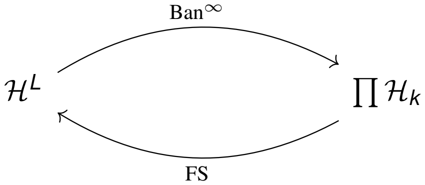

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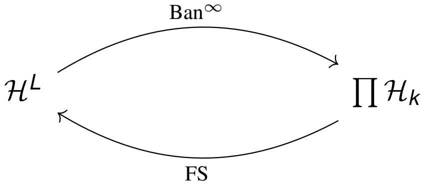

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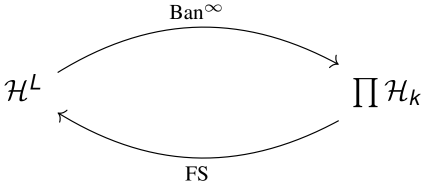

Question : to which extent, Ban^∞ and *FS* preserve geometry ?

Remark : We understand Ban^∞ much better than *FS*.

Natural metric structures on $\mathcal{H}^L$ and $\mathcal{H}_k$

We fix $p \in [1, +\infty[$, let H_0, H_1 Hermitian norms on $H^0(X, L^k)$

let $A \in \mathrm{End}^h(H^0(X, L^k))$ be such that $\langle \cdot, \cdot \rangle_{H_1} = \langle A \cdot, \cdot \rangle_{H_0}$

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$$d_p(H_0, H_1) := \sqrt[p]{\frac{\mathrm{Tr}\Big[|\log A|^p\Big]}{\dim V}}$$

For $N = \sum N_k$, $N = \sum N'_k$, we let

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Mabuchi : distance on $\mathcal{H}^L$, for a $\mathcal{C}^1$-path $\gamma = h^L_t \in \mathcal{H}^L$, $t \in [0, 1]$

$$\mathrm{len}_p(\gamma) = \int_0^1 \sqrt[p]{\int_X \left| \frac{\dot{h}^L_t}{h^L_t} \right|^p \cdot \frac{c_1(L, h^L_t)^n}{c_1(L)^n}}$$

$$d_p(h^L_0, h^L_1) := \inf_\gamma \mathrm{len}_p(\gamma).$$

Isometric properties of Ban^{∞} and *FS*

Theorem (Phong-Sturm, 2006, Chen-Sun, 2009, Berndtsson, 2012, Darvas-Lu-Rubinstein 2020)

Ban^{∞} is an isometry

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Conclusion : pluripotential theory $\rightsquigarrow$ study of filtrations

Proof of the main result

Main steps of the proof

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Let N_k norm on $H^0(X, L^k)$, then over $H^0(X, L^{kl})$:

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$\forall \epsilon > 0$, $\exists k_0$, so that $\forall k \geq k_0$, $N_{kl} \leq \exp(\epsilon kl) \cdot \mathrm{Ban}^{\infty}_{kl}(FS(N_k)^{\frac{1}{k}})$.
By Dini, $FS(N)$ continuous $\Rightarrow$ $FS(N_k)^{\frac{1}{k}}$ converge uniformly $\square$

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$$H^0(X, L^k)^{\otimes l} \xrightarrow{\mathrm{Sym}} \mathrm{Sym}^l(H^0(X, L^k)) = H^0(\mathbb{P}(H^0(X, L^k)^*), \mathscr{O}(l))$$

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Apply semiclassical OT extension theorem for the embedding

$$\mathbb{P}(V^*) \to \mathbb{P}((\mathbb{C}^l)^*)$$

Thank you！