Quantizations on groups

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Conference: Quantization in Geometry
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Plan

1. Kohn-Nirenberg and Wick quantizations* on $\mathbb{R}^n$ and on groups G.
2. The Hörmander pseudo-differential calculus on $\mathbb{R}^n$,
   Gårding inequality.
3. Pseudo-differential calculi for G compact or nilpotent Lie groups,
   Gårding inequality and applications.

* Quantization = symbol $\rightsquigarrow$ operator

Kohn-Nirenberg quantization on $\mathbb{R}^n$

Definition ($\mathrm{Op}^{\mathrm{KN}}\sigma$, $\sigma \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$)

$$(\mathrm{Op}^{\mathrm{KN}}\sigma) f(x) = \int_{\mathbb{R}^n} e^{2i\pi x\xi} \sigma(x,\xi) \widehat{f}(\xi)\, d\xi, \qquad f \in \mathscr{S}(\mathbb{R}^n),\ x \in \mathbb{R}^n,$$

$$\text{where } \mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\xi} f(x)\, dx.$$

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Convolution kernel $\kappa_x = \mathscr{F}^{-1}\sigma(x,\cdot)$

$$(\mathrm{Op}^{\mathrm{KN}}\sigma)f(x) = f * \kappa_x(x), \qquad f \in \mathscr{S}(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

The integral kernel of $\mathrm{Op}^{\mathrm{KN}}\sigma$ is $(x,y) \mapsto \kappa_x(x-y)$, hence

$$\|\mathrm{Op}^{\mathrm{KN}}\sigma\|_{\mathrm{HS}(L^2(\mathbb{R}^n))} = \|\kappa\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = \|\sigma\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)},$$

Hence, $\mathrm{Op}^{\mathrm{KN}} : L^2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathrm{HS}(L^2(\mathbb{R}^n))$ unitary and surjective.

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Remark: also on $\mathbb{T}^n$ using Fourier series…

Formal Kohn-Nirenberg Quantization on groups

M. Taylor '84

The definition of $\mathrm{Op}^{\mathrm{KN}}$ extends to groups where the Plancherel theorem based on the group Fourier transform (representation theory) holds.

Formal definition of $\mathrm{Op}^{\mathrm{KN}}\sigma$

$$(\mathrm{Op}^{\mathrm{KN}}\sigma)f(x) = \int_{\widehat{G}} \mathrm{tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right) d\mu(\pi), \qquad f \in C_c(G),\ x \in G.$$

For example, on the torus,

$$(\mathrm{Op}^{\mathrm{KN}}\sigma)f(x) = \sum_{\ell \in \mathbb{Z}^n} e^{2i\pi \ell x}\sigma(x,\ell)\,\widehat{f}(\ell), \qquad f \in C_c(\mathbb{T}^n),\ x \in \mathbb{T}^n.$$

Group Fourier transform and Plancherel theorem

Fourier transform on a locally compact group G

$$\mathscr{F}f(\pi) = \widehat{f}(\pi) = \int_G f(x)\,\pi(x)^*\,dx, \quad \pi \in \mathrm{Rep}\,G, f \in L^1(G).$$

Note $\widehat{f}(\pi) \in \mathscr{L}(\mathscr{H}_\pi)$ with $\|\widehat{f}(\pi)\|_{\mathscr{L}(\mathscr{H}_\pi)} \leq \|f\|_{L^1(G)}$ when π unitary.

Plancherel theorem (Dixmier, '60's)

- Hyp: G locally compact, unimodular, type I.
- $\widehat{G}$= set of unitary irreducible representation modulo equivalence.
- $\exists!$ Plancherel measure μ, $\quad \|f\|^2_{L^2(G)} = \int_{\widehat{G}} \|\widehat{f}(\pi)\|^2_{\mathrm{HS}(\mathscr{H}_\pi)} d\mu(\pi).$
- $T \in \mathscr{L}(L^2(G))$ left-invariant $\Longleftrightarrow Tf = \mathscr{F}^{-1}(\sigma\widehat{f}), \sigma \in L^\infty(\widehat{G})$.

  $$\|T\|_{\mathscr{L}(L^2(G))} = \|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathscr{H}_\pi)}, \quad \sigma = \{\sigma(\pi) \in \mathscr{L}(\mathscr{H}_\pi), \pi \in \widehat{G}\}.$$

Kohn-Nirenberg Quantization on groups

Op^{KN} *via the convolution kernel*

If $\sigma \in L^2(G \times \widehat{G})$, i.e. $\sigma(x,\pi) = \widehat{\kappa}_x(\pi)$ with $\kappa \in L^2(G \times G)$ or $C(G, \mathscr{F}L^1(G))$, then we define

$$(\mathrm{Op}^{KN}\sigma) f(x) = f * \kappa_x(x), \qquad x \in G,\ f \in C_c(G).$$

Integral kernel $(x,y) \mapsto \kappa_x(y^{-1}x)$.
Hence, $\mathrm{Op}^{KN} : L^2(G \times \widehat{G}) \to \mathrm{HS}(L^2(G))$ unitary and surjective.

Op^{KN} *via the symbol (M. Taylor '84)*

Assume that the inversion formula holds for 'enough' functions,

$$f(x) = \int_{\widehat{G}} \mathrm{tr}\left(\pi(x)\widehat{f}(\pi)\right) d\mu(\pi), \quad x \in G.$$

Then $(\mathrm{Op}^{KN}\sigma) f(x) = \int_{\widehat{G}} \mathrm{tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right) d\mu(\pi)$.

Wick quantization on $\mathbb{R}^n$

Generalised Bargmann transform

Fixing $a \in \mathscr{S}(\mathbb{R}^n)$ with $\|a\|_{L^2(\mathbb{R}^n)} = 1$, we set for $f \in L^2(\mathbb{R}^n)$

$$\mathscr{B}_a(f)(x,\xi) := \mathscr{F}(f\,a(\cdot - x))(\xi) = \int_{\mathbb{R}^n} f(y)\,a(y-x)\,e^{-2i\pi y\xi}\,dy, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

$\mathscr{B} = \mathscr{B}_a$ unitary transformation $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

Gaussian $a(x) = \pi^{-\frac{d}{2}} \mathrm{e}^{-\frac{|x|^2}{2}} \rightsquigarrow$ Bargmann transform.

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Wick quantization of $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$

$$\mathrm{Op}^{\mathrm{Wick}}(\sigma)f := \mathscr{B}^*\left(\sigma\,\mathscr{B}(f)\right), \qquad f \in L^2(\mathbb{R}^n).$$

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Advantages: L^2-boundedness, preserves adjoint and positivity

$$\|\mathrm{Op}^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \le \|\sigma\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}, \qquad \mathrm{Op}^{\mathrm{Wick}}(\sigma)^* = \mathrm{Op}^{\mathrm{Wick}}(\bar\sigma),$$

$$\sigma(x,\xi) \ge 0 \Longrightarrow (\mathrm{Op}^{\mathrm{Wick}}(\sigma) f, f)_{L^2(\mathbb{R}^n)} = (\sigma \mathscr{B} f, \mathscr{B} f)_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \ge 0.$$

Wick quantization on G (loc. comp., unimodular, type I)

Generalised Bargmann transform

Fixing $a \in C(G)$ bounded with $\|a\|_{L^2(\mathbb{R}^n)} = 1$, we set for $f \in L^2(G)$

$$\mathscr{B}_a(f)(x,\pi) := \mathscr{F}(f\, a(\cdot x^{-1}))(\pi), \qquad (x,\pi) \in G \times \widehat{G}.$$

$\mathscr{B} = \mathscr{B}_a$ unitary transformation $L^2(G) \to L^2(G \times \widehat{G})$.

Wick quantization of $\sigma \in L^\infty(G \times \widehat{G})$

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$$\|\mathrm{Op}^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))} \leq \|\sigma\|_{L^\infty(G\times\widehat{G})} =: \sup_{(x,\pi)\in G\times\widehat{G}} \|\sigma(x,\pi)\|_{\mathscr{L}(\mathscr{H}_\pi)},$$

$$\mathrm{Op}^{\mathrm{Wick}}(\sigma)^* = \mathrm{Op}^{\mathrm{Wick}}(\sigma^*),$$

$$\sigma(x,\pi) \geq 0 \implies (\mathrm{Op}^{\mathrm{Wick}}(\sigma) f, f)_{L^2(G)} = (\sigma \mathscr{B} f, \mathscr{B} f)_{L^2(G\times\widehat{G})} \geq 0.$$

Link between $\mathrm{Op}^{\mathrm{KN}}$ *and* $\mathrm{Op}^{\mathrm{Wick}}$

Link via convolution kernel - when it makes sense...

Let $\sigma \in L^\infty(G \times \widehat{G})$ with a convolution kernel $\kappa \in C(G, L^1(G)$. Then

$$\mathrm{Op}^{\mathrm{Wick}}(\sigma) f(x) = f * \kappa_x^{\mathrm{Wick}}(x), \qquad f \in \mathscr{C}_c(G),\, x \in G,$$

where

$$\begin{aligned}\kappa_x^{\mathrm{Wick}}(w) &= \int_G a(w^{-1} x z^{-1}) \bar{a}(x z^{-1}) \kappa_z(w)\, dz \\ &= \int_G a(w^{-1} z') \bar{a}(z') \kappa_{z'^{-1} x}(w)\, dz'.\end{aligned}$$

Application

Gårding inequalities within pseudo-differential calculi à la Hörmander.

Gårding inequality on $\mathbb{R}^n$

Hörmander classes $S^m(\mathbb{R}^n)$ and $\Psi^m(\mathbb{R}^n)$

$$\forall \alpha,\beta\in\mathbb{N}_0^n \quad \exists C_{\alpha,\beta}>0 \quad \forall (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n \quad |\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)|\le C_{\alpha,\beta}(1+|\xi|^2)^{\frac{m-|\alpha|}{2}}.$$

$\rightsquigarrow S^m(\mathbb{R}^n)$, and $\Psi^m(\mathbb{R}^n):=\mathrm{Op}^{\mathrm{KN}}(S^m(\mathbb{R}^n))$.

Sharp strong Gårding inequality

If $\sigma\in S^m(\mathbb{R}^n)$, $m\in\mathbb{R}$, satisfies $\sigma(x,\xi)\ge c(1+|\xi|^2)^{m/2}$, for $c>0$, then $\exists C>0$ s.t.

$$\forall f\in\mathscr{S}(\mathbb{R}^n) \qquad \Re\left(\mathrm{Op}^{\mathrm{KN}}(\sigma)f,f\right)_{L^2}\ge -C\|f\|^2_{H^{\frac{m-1}{2}}}.$$

Generalisation to matrix-valued symbols, and
with modifications to (ρ,δ)-classes and to $\sigma\ge 0$.

Proof (Benedetto + Fermanian + F., '23)

Pseudo-diff. calculus: it suffices to show $m = 0$, so let $\sigma \in S^0(\mathbb{R}^n)$, $\sigma(x,\xi) \geq c$.

Link between $\mathrm{Op}^{\mathrm{Wick}}$ *and* $\mathrm{Op}^{\mathrm{KN}}$ *($a \in \mathscr{S}(\mathbb{R}^n)$, $\|a\|_{L^2(\mathbb{R}^n)} = 1$).*

$$\sigma \in S^0(\mathbb{R}^n) \implies \mathrm{Op}^{\mathrm{Wick}}(\sigma) - \mathrm{Op}^{\mathrm{KN}}(|a|^2 * \sigma) \in \Psi^{-1}(\mathbb{R}^n).$$

$$\begin{aligned}&\Re(\mathrm{Op}^{\mathrm{KN}}(\sigma)f, f)_{L^2} \\ &\quad \geq \underbrace{(\mathrm{Op}^{\mathrm{Wick}}(\sigma)f, f)_{L^2}}_{=(\sigma\mathscr{B}f,\mathscr{B}f)_{L^2} \geq c\|f\|_{L^2}} - \underbrace{\|\mathrm{Op}^{\mathrm{KN}}(\sigma - |a|^2 * \sigma)\|_{\mathscr{L}(L^2)}}_{\leq \|\sigma - |a|^2 * \sigma\|_{S^0, a_0, b_0}} \|f\|^2_{L^2(G)} - C\|f\|^2_{H^{-\frac{1}{2}}}.\end{aligned}$$

Approximation of the identity

If $\varphi_1 \in \mathscr{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi_1 = 1$, then $\varphi_t = t^{-n}\varphi(t^{-1}\cdot)$, $t > 0$, satisfies

$$\lim_{t \to 0} \|\sigma - \varphi_t * \sigma\|_{S^m, a_1, b_1} = 0.$$

Fix $a_1 \in \mathscr{S}(\mathbb{R}^n)$ with $\|a_1\|_{L^2} = 1$. Set $a_t := t^{-n/2} a(t^{-1}\cdot)$ for $t > 0$. Choose $a = a_t$ in the Wick quantization with $t > 0$ s.t. $\|\sigma - |a|^2 * \sigma\|_{S^0, a_0, b_0} \leq c$. $\square$

What is a pseudo-differential calculus?

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$\Psi^\infty(M)$ is a space of continuous operators $\mathcal{D}(M) \to \mathcal{D}(M)$

- that is filtered ($m \le m' \Rightarrow \Psi^m(M) \subset \Psi^{m'}(M)$),
- stable by composition $\Psi^{m_1} \times \Psi^{m_2} \to \Psi^{m_1+m_2}$ and $* : \Psi^m(M) \to \Psi^m(M)$
- that contains a differential calculus and acting continuously on 'Sobolev-like spaces'.

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+ each $\Psi^m(M)$, $m \in \mathbb{R}$, is a Fréchet space:

- continuous inclusion, composition and $*$.
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Optional: symbol, asymptotic expansions.

Pseudo-differential calculus on a compact Lie group G

Plancherel theorem on G = Peter-Weyl theorem

$\widehat{G}$ is discrete and $\mu(\{\pi\}) = \dim \pi$: $\|f\|^2_{L^2(G)} = \sum_{\pi \in \widehat{G}} \dim \pi \|\widehat{f}(\pi)\|^2_{HS(\mathscr{H}_\pi)}$.

$\rightsquigarrow$ symbol $\sigma = \{\sigma(x,\pi) \in \mathscr{L}(\mathscr{H}_\pi) : (x,\pi) \in G \times \widehat{G}\}$

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Laplace-Beltrami operator

Fix an ONB $X_1, \ldots, X_n$ on $\mathfrak{g}$. $\quad \mathcal{L} := -X_1^2 - \ldots - X_n^2$.
If π irreducible representation of G, then $\pi(\mathcal{L}) = \lambda_\pi \text{Id}$.

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Hörmander classes $S^m(G)$ and $\Psi^m(G)$

$$\forall\alpha,\beta \quad \exists C>0 \quad \forall(x,\pi)\in G\times\widehat{G} \quad \|X_x^\beta\Delta^\alpha\sigma(x,\pi)\|_{\mathscr{L}(\mathscr{H}_\pi)} \le C(1+\lambda_\pi)^{\frac{m-|\alpha|}{2}}.$$

$\rightsquigarrow$ $S^m(G)$, and $\Psi^m(G) := \mathrm{Op}^{KN}(S^m(G))$, pseudo-diff. calculus *(F. JFA '15)*.
Difference operators Δ^α intrinsic *(F. JFA '15, '20)*
or implicitly $\Delta_q\widehat{\kappa} = \mathscr{F}(q\kappa)$ *(Ruzhansky+Turunen+Wirth, 2010-14)*.

What is this pseudo-differential calculus good for?

Link with Hörmander's (F. '15 and '20)

- $\Psi^\infty(G)$ coincides with the pseudo-differential calculus on $M = G$ defined via charts etc. Generalisation for $1 \geq \rho > \delta \geq 0$, $\rho \geq 1 - \delta$.
- Polyhomogeneous symbols in $S^m(G)$, and link with Hörmander's.

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Applications

- *Ruzhansky+Turunen+Wirth '14*: Global hypoellipticity of $X + c$ on $G = \mathrm{SU}(2)$ with $X \in \mathfrak{su}_2$, several values of the constant c.
- *Bambusi+Langella '22, Growth of Sobolev norms in quasi integrable quantum systems* $\quad i\partial_t \psi = (H_0 + V(t))\psi$ on M, $H_0 = \mathrm{Id} + \sum_j A_j^2$, $A_j \in \Psi^1$ + hyp that are satisfied on $M = G$ using *F. JFA '15.*
(To do: homogeneous spaces G/K)
- *Shao '23* uses *F. JFA '15* for a paradifferential calculus on G.

Wick quantization and Gårding inequality on G compact

Link between $\mathrm{Op}^{\mathrm{Wick}}$ *and* $\mathrm{Op}^{\mathrm{KN}}$

Assume $a \in C^\infty(G)$ with $\|a\|_{L^2(\mathbb{R}^n)} = 1$. If $\sigma \in S^0(G)$, then

$$\mathrm{Op}^{\mathrm{Wick}}(\sigma) - \mathrm{Op}^{\mathrm{KN}}(|a|^2 * \sigma) \in \Psi^{-1}(G).$$

Sharp strong Gårding inequality (Benedetto+Fermanian+F. '23)

If $\sigma \in S^m(G)$, $m \in \mathbb{R}$, satisfies $\sigma(x,\xi) \geq c(\mathrm{Id} + \lambda_\pi)^{m/2}$, for $c > 0$, then $\exists C > 0$ s.t.

$$\forall f \in C^\infty(\mathbb{R}^n) \qquad \Re\left(\mathrm{Op}^{\mathrm{KN}}(\sigma) f, f\right)_{L^2} \geq -C\|f\|^2_{H^{\frac{m-1}{2}}}.$$

+ generalisations.

Proof: Same ingredients as on $\mathbb{R}^n$ with $a = \sqrt{p_t}$, since the heat kernels p_t, $t > 0$, are an approximation of the identity.

Pseudo-differential calculus on a graded nilpotent Lie group G

G not compact, $\widehat{G}$ non-discrete.
However, $\widehat{G}$ and μ may be described by the orbit method (Kirillov).
Left-invariant differential calculus ≠ right-invariant!

Hörmander classes $S^m(G)$ and $\Psi^m(G)$, F.+Ruzhansky '16

$$\|\pi(\text{Id}+\mathscr{R})^{-\frac{m-\rho[\alpha]+\delta[\beta]+\gamma}{\nu}} X^\beta \Delta^\alpha \sigma(x,\pi)\pi(\text{Id}+\mathscr{R})^{\frac{\gamma}{\nu}}\|_{\mathscr{L}(\mathscr{H}_\pi)} \leq C_{\alpha,\beta,\gamma},$$

where $\mathscr{R}$ positive Rockland operator of homogeneous degree ν,
difference operators $\Delta^\alpha = \Delta_{x^\alpha}$.
$\rightsquigarrow S^m(G)$, and $\Psi^m(G) := \text{Op}^{\text{KN}}(S^m(G))$.
Pseudo-differential calculus on adapted Sobolev spaces $L^2_s(G)$.

Wick quantization and Gårding inequality on G graded

Link between $\mathrm{Op}^{\mathrm{Wick}}$ *and* $\mathrm{Op}^{\mathrm{KN}}$

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If $\sigma \in S^m(G)$, $m \in \mathbb{R}$, satisfies $\sigma(x,\xi) \geq c(\mathrm{Id} + \widehat{\mathscr{R}})^{m/\nu}$ for $c > 0$, then $\exists C > 0$ s.t.

$$\forall f \in \mathscr{S}(\mathbb{R}^n) \qquad \Re\left(\mathrm{Op}^{\mathrm{KN}}(\sigma) f, f\right)_{L^2} \geq -C \|f\|^2_{L^2_{\frac{m-1}{2}}}.$$

+ generalisations.

Proof: 'Almost' same ingredients as on $\mathbb{R}^n$.

Remark: in *Benedetto+Fermanian+F. '23*, also the semi-classical case.

Applications: quantum limit as operator-valued measures

Fermanian + F. '20: Micro-local Defect Measures on G

Given $(f_j)_{j\in\mathbb{N}}$, $\|f\|_{L^2(G)}=1$, $f_j \rightharpoonup 0$, $\exists\ (j_k)$, $\Gamma d\gamma$ s.t. $\forall A\in\Psi^0_{cl}(G)$, princ A$=\sigma_0$

$$\left(A f_j, f_j\right)_{L^2} \longrightarrow_{j=j_k\to\infty} \int_{G\times(\widehat{G}\setminus\{1\})/\mathbb{R}^+} \mathrm{tr}\left(\pi(x)\sigma_0(x,\dot\pi)\Gamma(x,\dot\pi)\right) d\gamma(x,\dot\pi).$$

Fermanian + F. '19: Semi-classical calculus on G

$$\mathrm{Op}^{\mathrm{KN}}_\varepsilon(\sigma) = \mathrm{Op}\left(\sigma(\,\cdot\,,\delta_\varepsilon\,\cdot\,)\right),\ \varepsilon>0, \quad \text{with } \sigma(x,\pi)=\widehat{\kappa}_x(\pi),\ \kappa\in C_c^\infty(G,\mathscr{S}(G)).$$

Fermanian + F. '21: semi-classical measures

Given $(f_\varepsilon)_{\varepsilon\in(0,1]}$, $\|f_\varepsilon\|_{L^2}=1$, $\exists\ (\varepsilon_k)$, $\Gamma d\gamma$ s.t.

$$\left(\mathrm{Op}^{\mathrm{KN}}_\varepsilon(\sigma)\, f_\varepsilon, f_\varepsilon\right)_{L^2} \longrightarrow_{\varepsilon=\varepsilon_k\to 0} \int_{G\times\widehat{G}} \mathrm{tr}\left(\pi(x)\sigma(x,\pi)\Gamma(x,\pi)\right) d\gamma(x,\pi).$$

+ Quantum evolution of the sub-laplacian on H-type groups.

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Selection of published works

- *Fermanian + F. + Flynn '21*: Geometric invariance of the semi-classical calculus on graded nilpotent Lie groups, *JGEA*.
- *F. '22:* Asymptotics and zeta functions on nilmanifolds, *JMPA*.
- *Fermanian + F. + Flynn '22*: Some remarks on semi-classical analysis on 2-step nilmanifolds, *Proceedings IQM22*.

Work in progress on regular subRiemannian manifolds

- *Fermanian + F. + Flynn*: quantization with symbol $\sigma(x,\pi)$, $x \in G$, $\pi \in \widehat{G}_x$, pseudo-diff. calculus.
- *F. + Mikkelsen*: Weyl law estimates of subLaplacians.

Longer term: *Fermanian + F.* : quantum evolution and QE of subLaplacians?

Thank you for your attention.