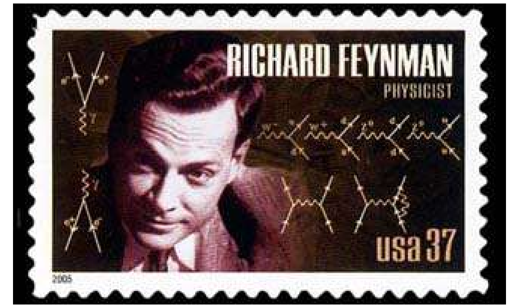
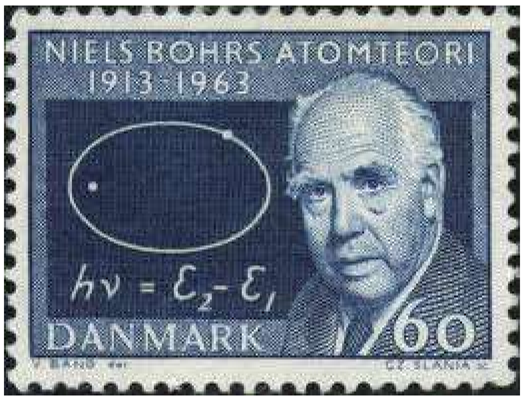
Quantization

I think I can safely say that nobody understands quantum mechanics.
Richard Feynman

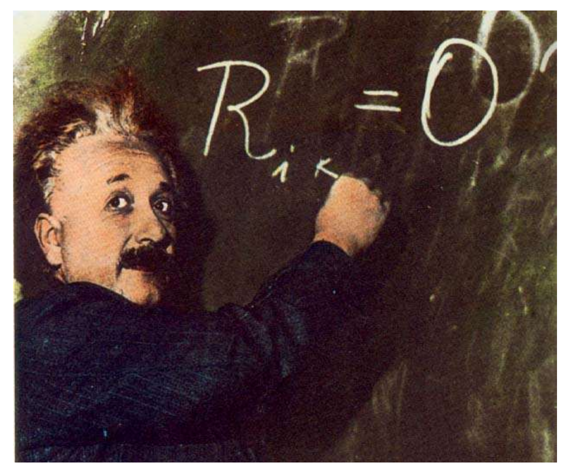




Anyone who is not shocked by quantum theory has not understood a single word.
Niels Bohr

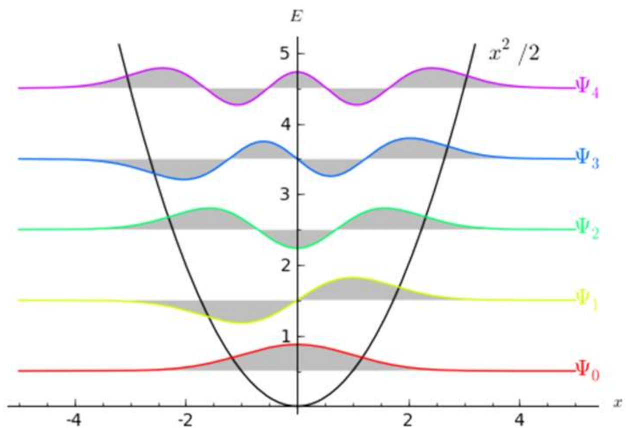
Very interesting theory -- it makes no sense at all.
Groucho Marx

Gott würfelt nicht!
Albert Einstein



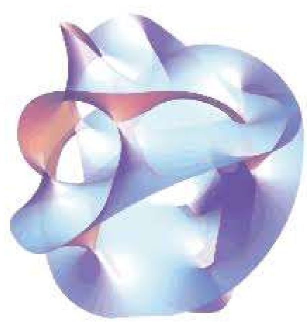
The more success the quantum theory has the sillier it looks.

Geometry	Quantum algebra
symplectic manifold (M, ω) $\rightsquigarrow$	$\mathcal{H}$ (Hilbert space)
algebra of functions $\rightsquigarrow$	$\mathcal{A}$ (algebra of operators)
Lagrangian submanifolds $L \subset M$ $\rightsquigarrow$	vectors $Z \in \mathcal{H}$



Mirror symmetry:

Complex (B-model)		Symplectic (A-model)
variations of Hodge structure	$\longleftrightarrow$	quantum cohomology, curve counting
$D^b\mathrm{Coh}(X)$	$\longleftrightarrow$	$D^b\mathrm{Fuk}(Y)$



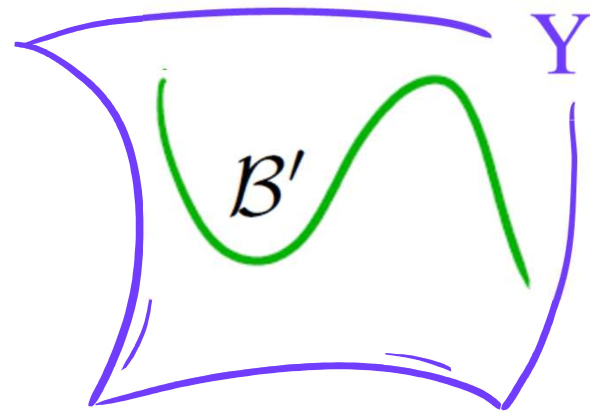
M.Kontsevich

Brane quantization:

S.G, E.Witten ('08)

A-model of $Y = M_{\mathbb{C}}$:

$$\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$$
$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$$



$\mathcal{B}'$ = Lagrangian A-brane supported on $M \subset Y$

$\mathcal{B}_{cc}$ = coisotropic A-brane supported on Y and endowed with a unitary line bundle $\mathcal{L}$ with a connection of curvature $F = \mathrm{Re}\,\Omega$

A.Kapustin, D.Orlov

Geometry

Quantum algebra

symplectic manifolds, non-compact moduli spaces, ...

interesting algebras, representations, modular categories, ...

Geometry

Quantum algebra

Quantization

algebra $\mathcal{A}$, ...

A-model

X

MTC[X]

(modular) tensor category

B-model

Geometry

Quantum algebra

Coulomb branch

X

non-compact hyper-Kahler

$I \quad J \quad K$

$\omega_I \quad \omega_J \quad \omega_K$

Quantization

algebra $\mathcal{A}$, ...

A-model

MTC[X]

(modular) tensor category

B-model

Geometry

$X = T^*\mathbb{C}\mathbf{P}^1$

$X = T^*\mathbb{C}\mathbf{P}^n$

$X = T^*Gr(n,N)$

$X = \mathrm{Hilb}^n(\mathbb{C}^2)$

$X = \mathcal{M}_H(G,C)$

Lee-Weinberg-Yi

Taubian-Calabi

ALE, ALF, ALG, ALG* ...

Quantum algebra

Quantization

algebra $\mathcal{A}$, ...

A-model

MTC[X]

(modular) tensor category

B-model

Branes and DAHA Representations

Sergei Gukov[1] Peter Koroteev[2,3] Satoshi Nawata[4] Du Pei[5] Ingmar Saberi[6]




A-model

Rozansky-Witten geometry of Coulomb branches and logarithmic knot invariants

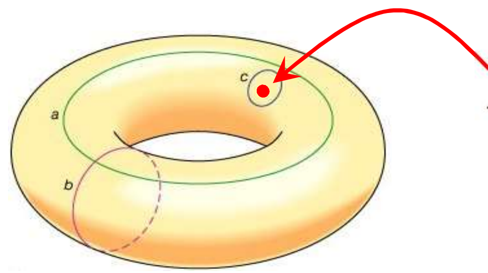
Sergei Gukov[1] Po-Shen Hsin[1] Hiraku Nakajima[2] Sunghyuk Park[3] Du Pei[4] Nikita Sopenko[1]





B-model

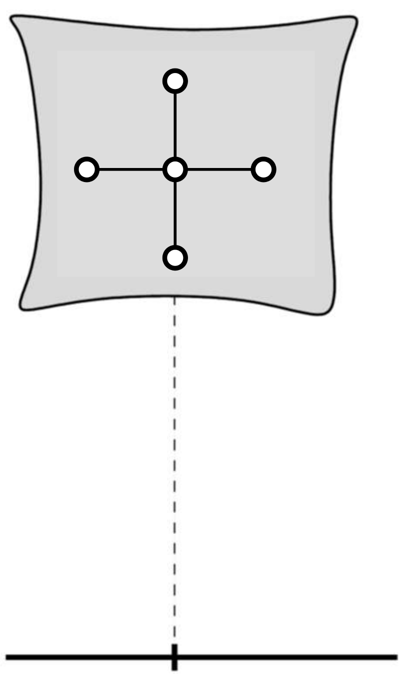
Example:



$$V \sim \begin{pmatrix} t^{-2} & 0 \\ 0 & t^{2} \end{pmatrix}$$

$$Y \quad : \qquad x^2 + y^2 + z^2 - xyz \;=\; \operatorname{tr} V + 2$$

Deformation (smoothing) of $\dfrac{\mathbb{C}^* \times \mathbb{C}^*}{\mathbb{Z}_2}$

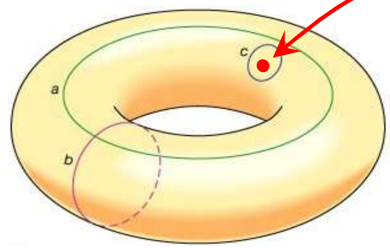


$$\begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

W.Goldman
T.Hausel
E.Frenkel, E.Witten

2010 Takagi Lectures

Example:



$$V \sim \begin{pmatrix} t^{-2} & 0 \\ 0 & t^{2} \end{pmatrix}$$

$$Y \;:\; \qquad x^2 + y^2 + z^2 - xyz \;=\; \operatorname{tr} V + 2$$

Brane quantization:

$$q \;=\; e^{2\pi i \hbar} \qquad\qquad \hbar = |\hbar| e^{i\theta} \qquad\qquad \Omega = \frac{1}{2\pi\hbar}\frac{dx \wedge dy}{xy - 2z}$$

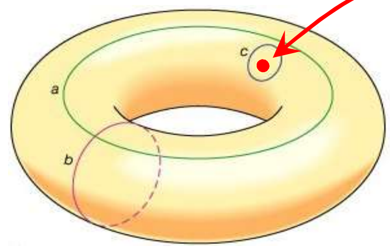
$\mathcal{B}_{cc}$

$$\mathrm{Re}\;\Omega = \frac{1}{|\hbar|}(\omega_I \cos\theta - \omega_K \sin\theta)\ ,$$

A-model

$$\mathrm{Im}\;\Omega = -\frac{1}{|\hbar|}(\omega_I \sin\theta + \omega_K \cos\theta)$$

Example:



$$V \sim \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix}$$

$$Y \;:\qquad x^2 + y^2 + z^2 - xyz \;=\; \mathrm{tr}\, V + 2$$

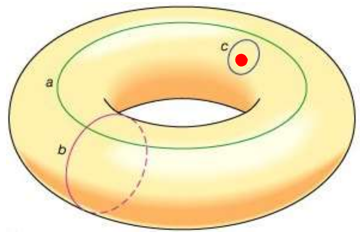
$$\mathcal{A} \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \;=\; \text{spherical DAHA}$$

A.Oblomkov

$$\mathcal{B}_{cc} \longrightarrow \quad \mathrm{Re}\;\Omega = \frac{1}{|\hbar|}(\omega_I \cos\theta - \omega_K \sin\theta)\ ,$$

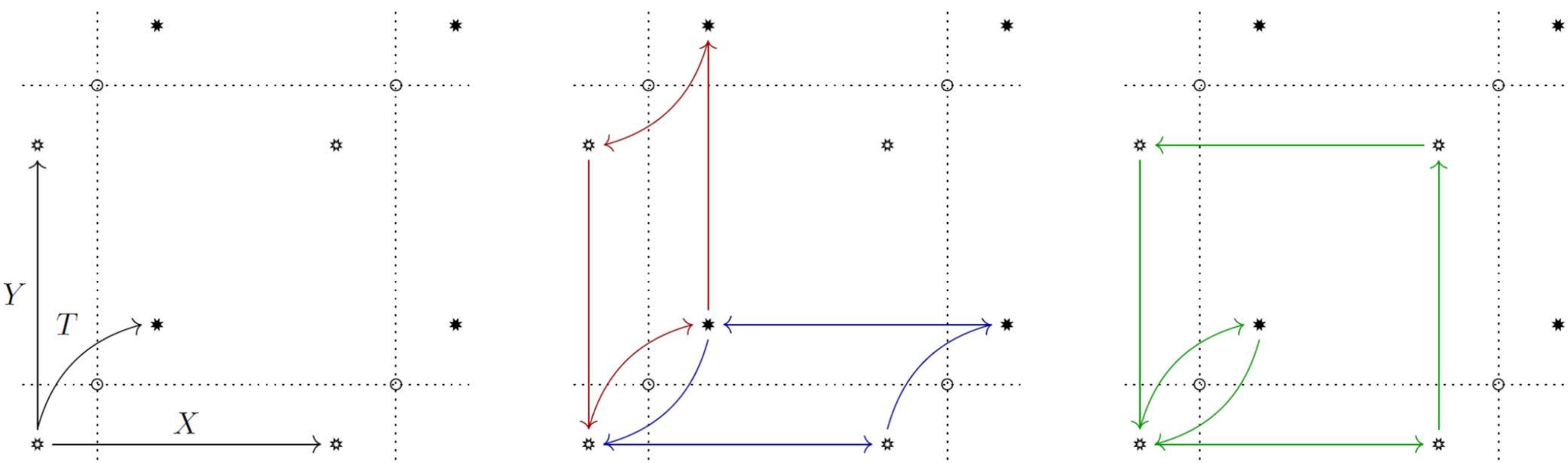
$$\text{A-model} \longrightarrow \quad \mathrm{Im}\;\Omega = -\frac{1}{|\hbar|}(\omega_I \sin\theta + \omega_K \cos\theta)$$

Example:



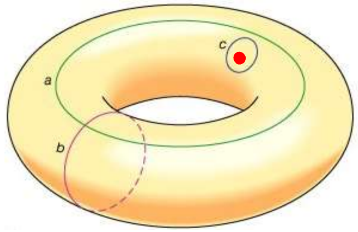
Elliptic braid group (a.k.a. double affine braid group):

$$\pi_1^{\mathrm{orb}}(\{E \setminus 0\}/\mathbb{Z}_2)$$



$$TY^{-1}T = Y \qquad Y^{-1}X^{-1}YXT^2 = 1$$

$$TXT = X^{-1}$$

Example: 

Elliptic braid group (a.k.a. double affine braid group):

$$\pi_1^{\mathrm{orb}}(\{E \setminus 0\}/\mathbb{Z}_2)$$

$$TY^{-1}T = Y \qquad Y^{-1}X^{-1}YXT^2 = 1$$

$$TXT = X^{-1}$$

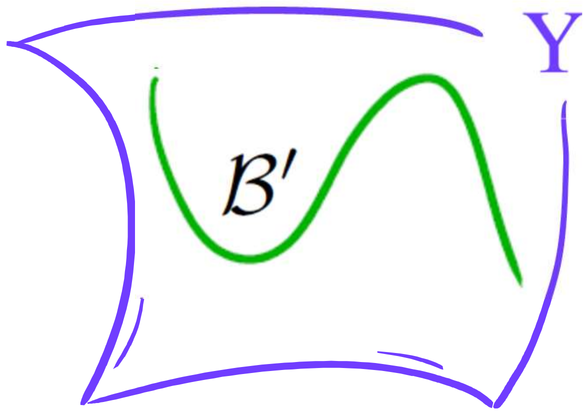
Central extension: $Y^{-1}X^{-1}YXT^2 = q^{-1}$

Double Affine Hecke Algebra (DAHA):

$$\mathbb{C}[B_{\mathrm{ell}}]/((T-t)(T+t^{-1}))$$

$$\mathcal{A} \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \;=\; \text{spherical DAHA}$$

A-branes $\mathcal{B}'$ $\longrightarrow$ Representations of (spherical) DAHA



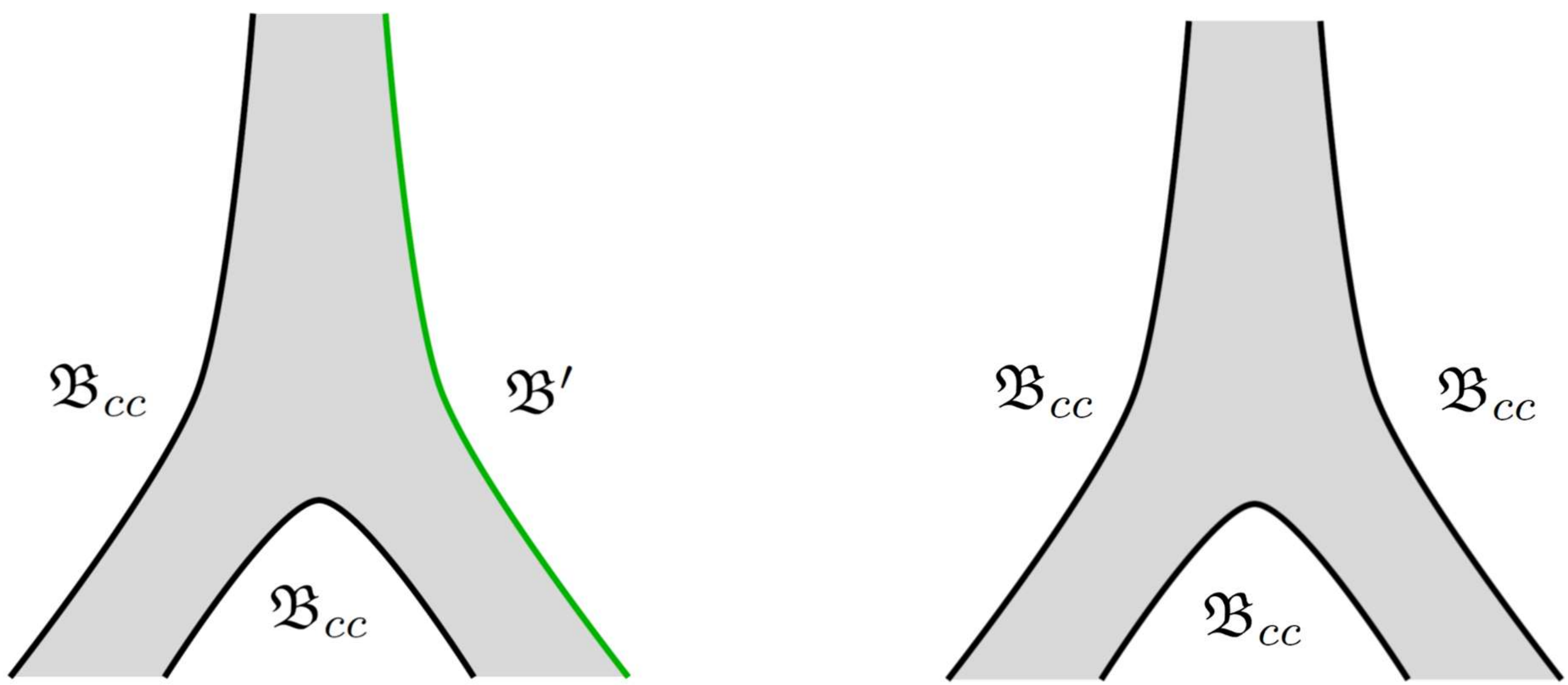
$$\mathcal{H} \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$$

S.G., P.Koroteev, S.Nawata, D.Pei, I.Saberi

cf. M.Varagnolo, E.Vasserot
E.Gorsky, A.Oblomkov, J.Rasmussen, V.Shende
A.Braverman, P.Etingof, M.Finkelberg, H.Nakajima, D.Yamakawa
:

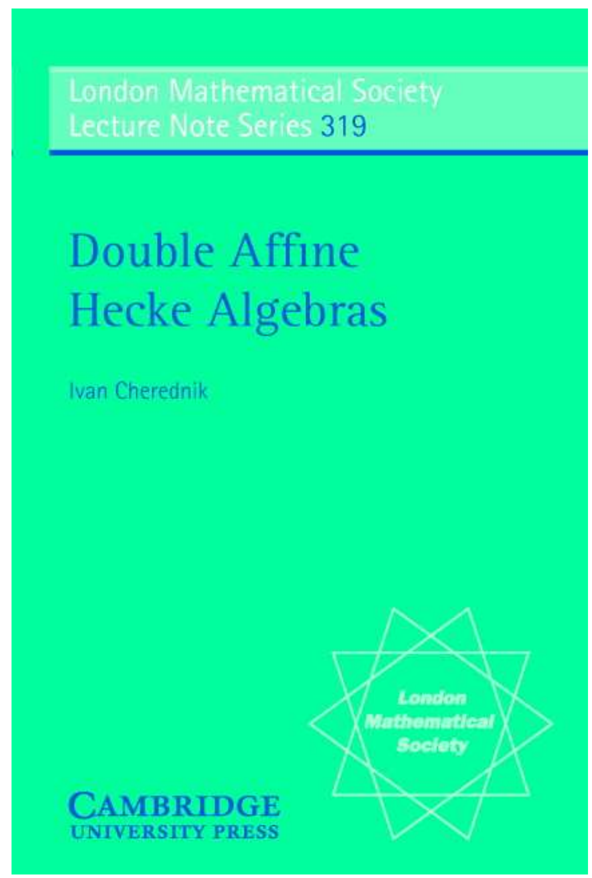
$$\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \text{spherical DAHA}$$

A-branes $\mathcal{B}'$ $\longrightarrow$ Representations of (spherical) DAHA



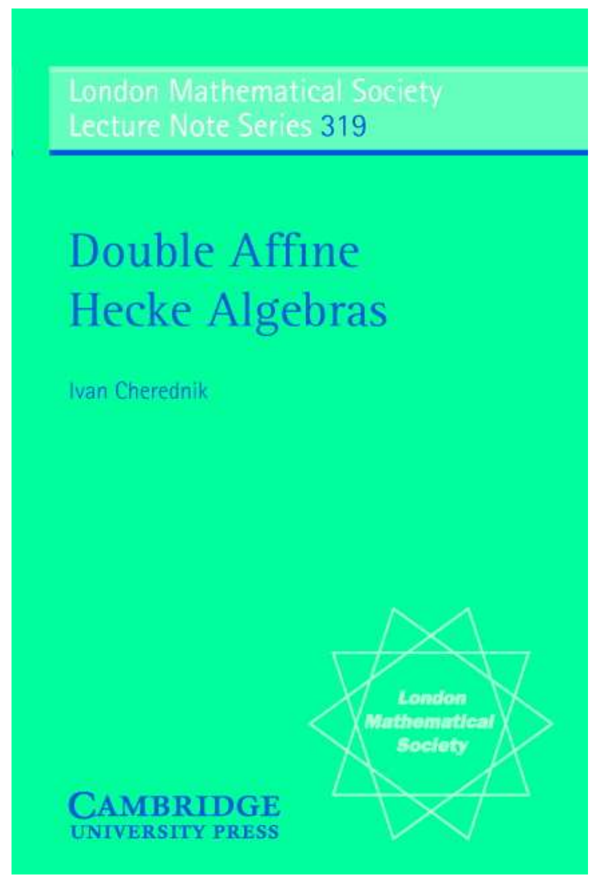
$$\mathcal{A} \;=\; \mathrm{Hom}(\mathcal{B}_{cc},\mathcal{B}_{cc}) \;=\; \text{spherical DAHA}$$

A-branes $\mathcal{B}'$ $\longrightarrow$ Representations of (spherical) DAHA



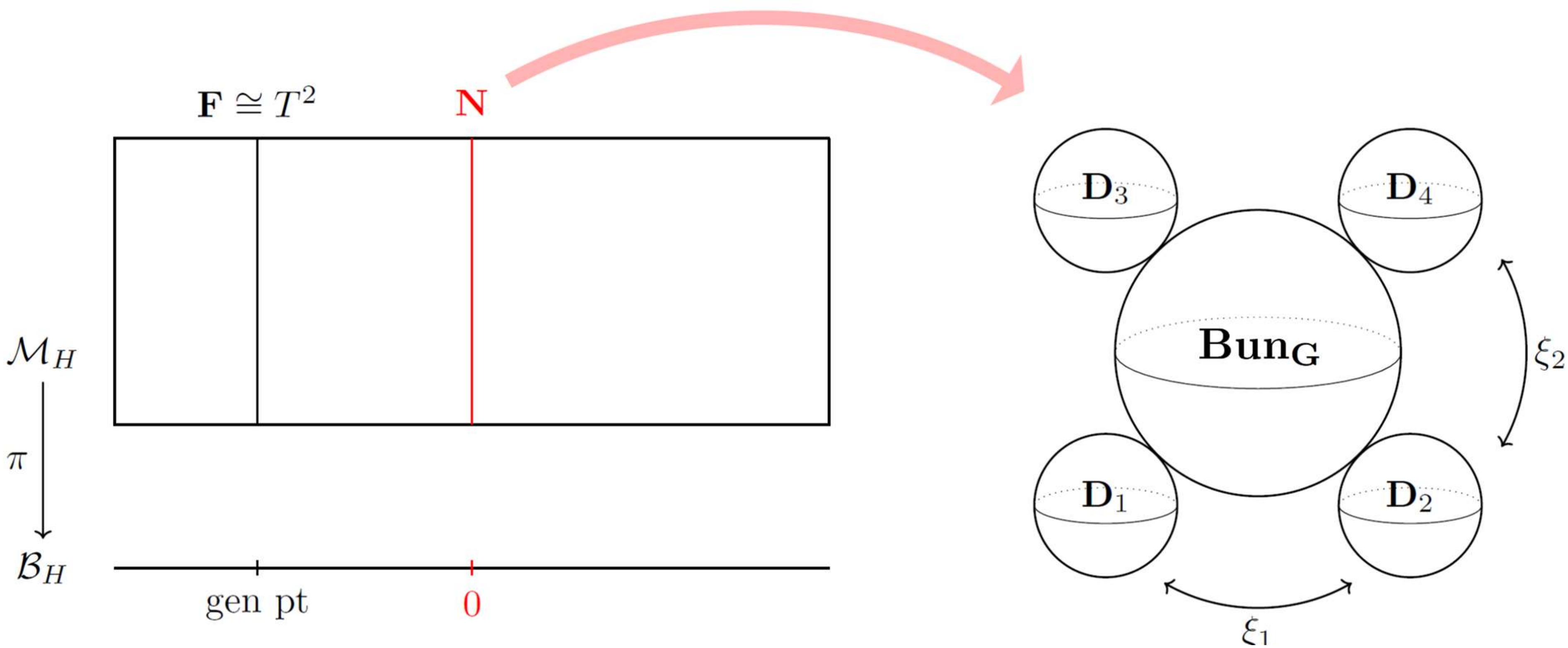
$$\mathcal{A} \; = \; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \; = \; \text{spherical DAHA}$$

A-branes $\mathcal{B}'$ $\longrightarrow$ Representations of (spherical) DAHA



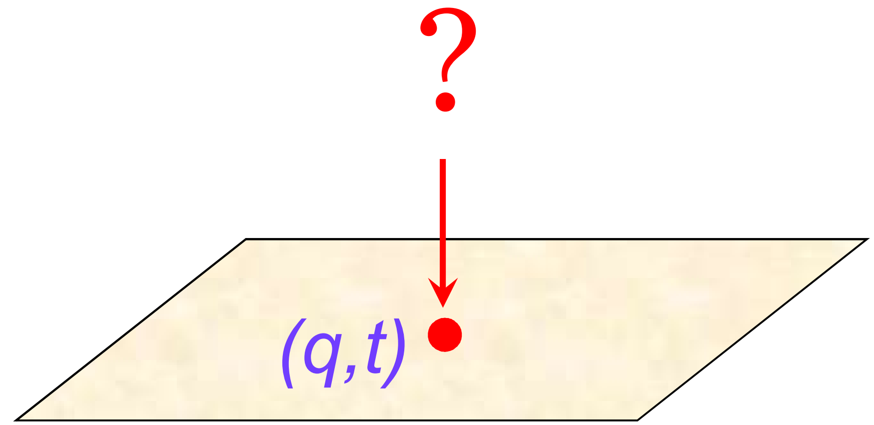
$$\mathcal{A} \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \;=\; \text{spherical DAHA}$$

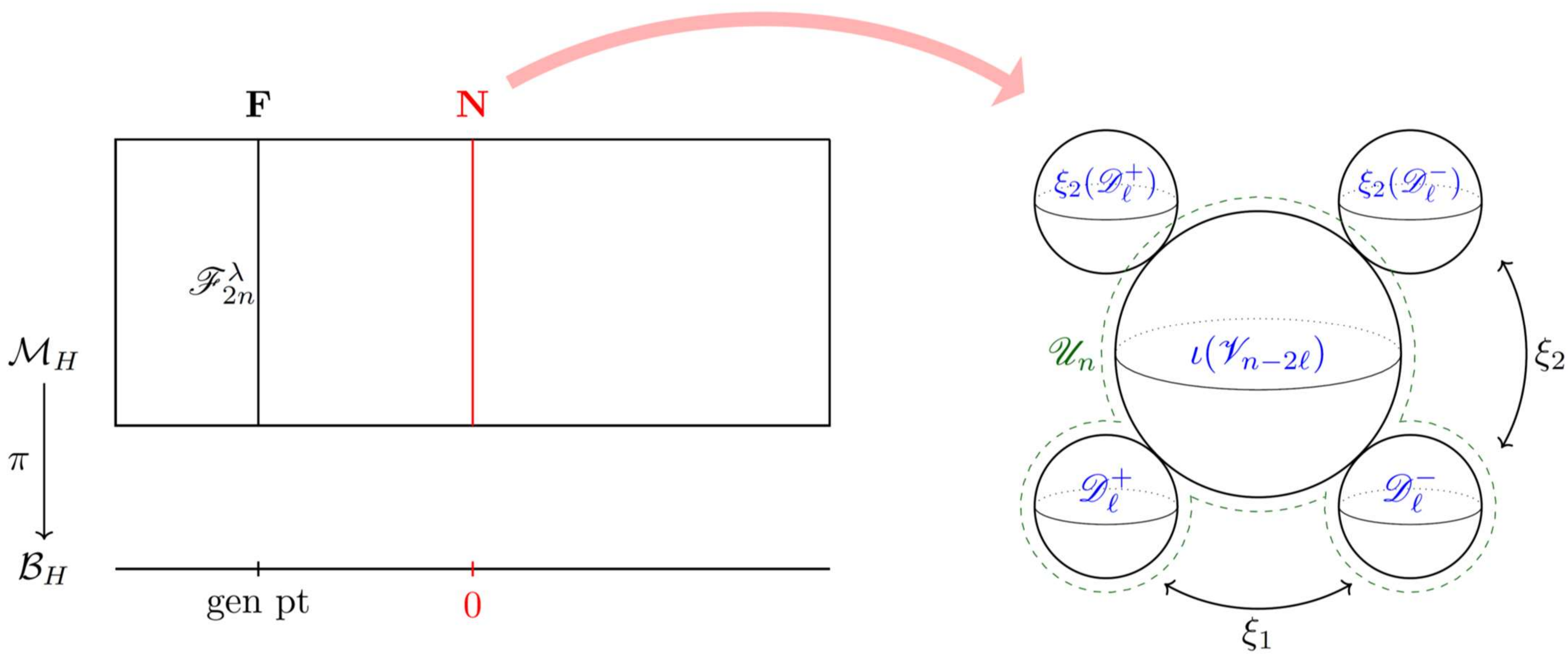
A-branes $\mathcal{B}'$ $\longrightarrow$ Representations of (spherical) DAHA



Interesting branes $\mathcal{B}'$:

- generic fiber $\mathbf{F}$
- $\mathbf{Bun_G}$
- exceptional divisors $\mathbf{D}_i$
- non-trivial extensions $\mathscr{U}$



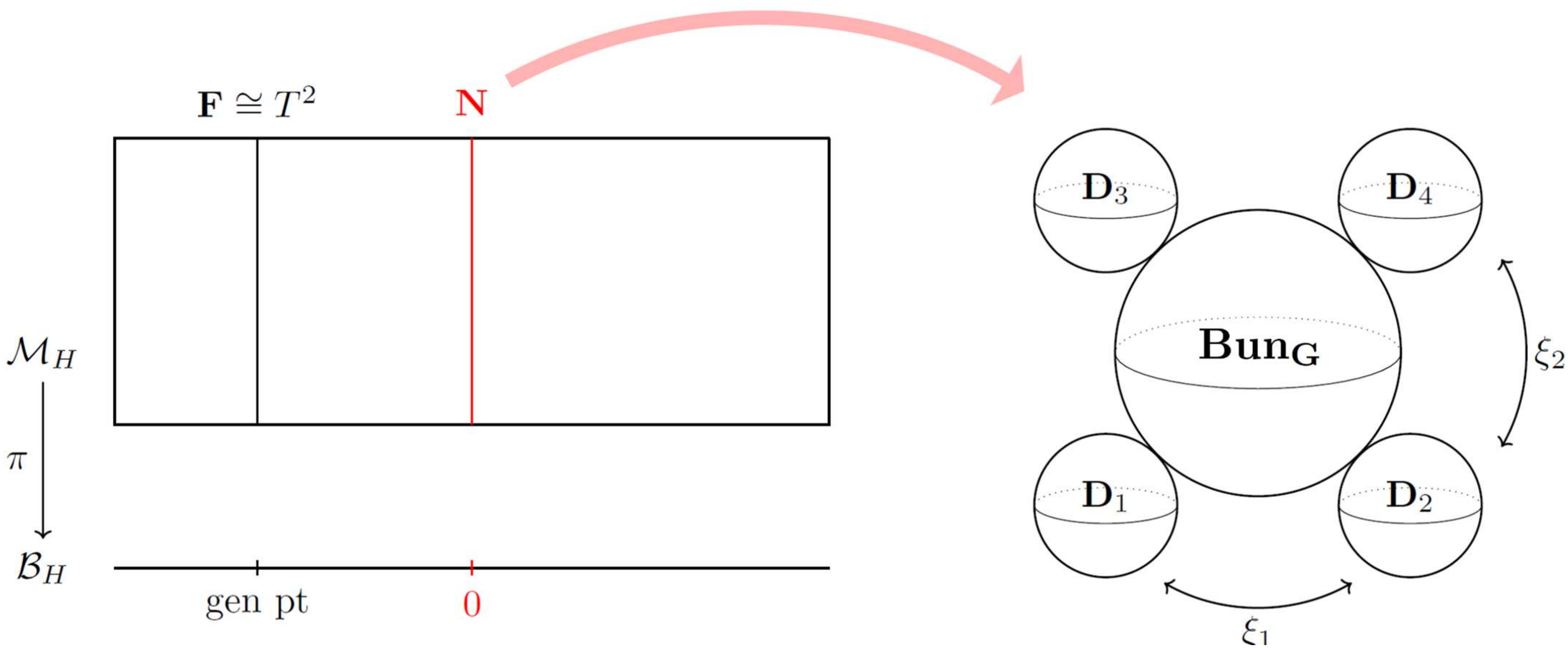


$\mathcal{B}'$ = generic fiber (B,A,A) brane

parameters $\lambda \in \mathbb{C}^\times \times \mathbb{C}^\times$

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{F}}^{\lambda}) = \int_{\mathbf{F}} \frac{\omega_I}{2\pi\hbar} = \frac{1}{\hbar} \Rightarrow q = e^{2\pi i/m}, \quad m \in \mathbb{Z}_{>0}$$

t = unrestricted



$\mathcal{B}' = \mathbf{Bun_G}$

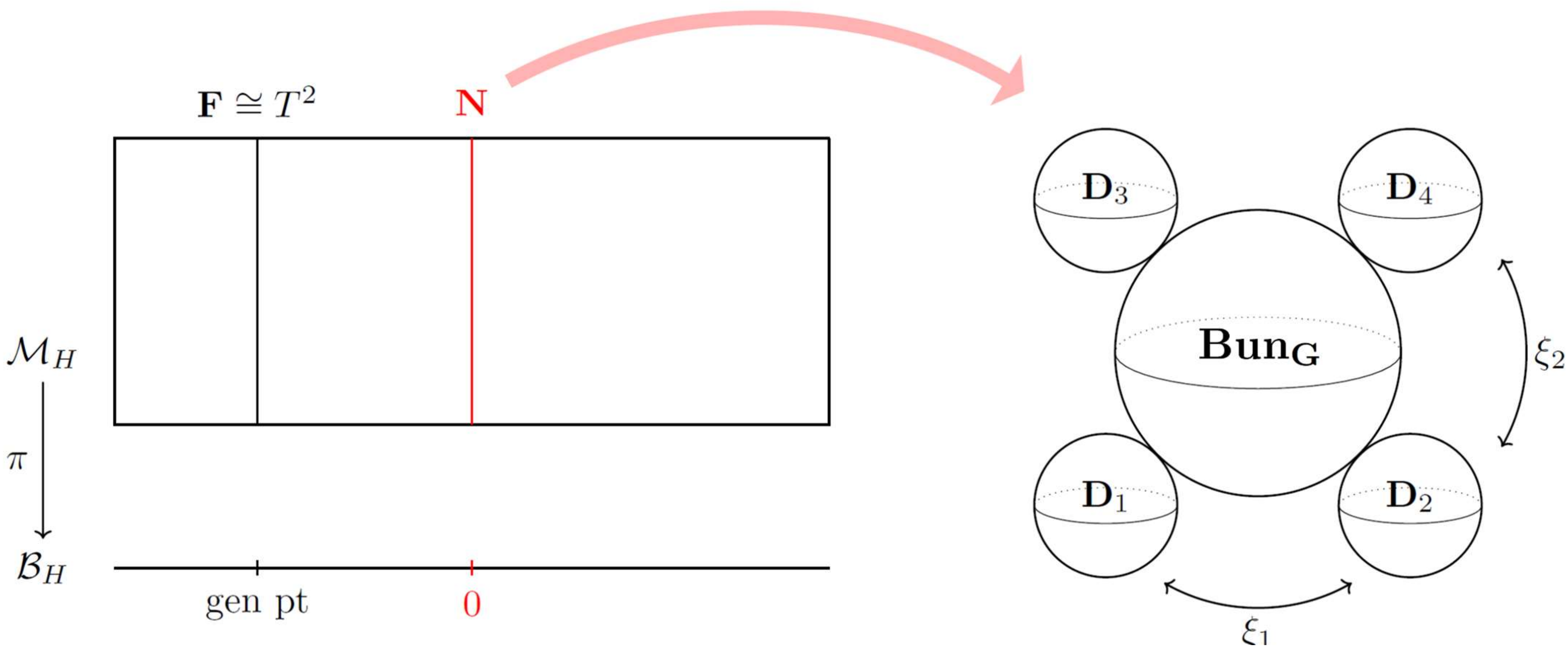
(B,A,A) brane

no additional parameters

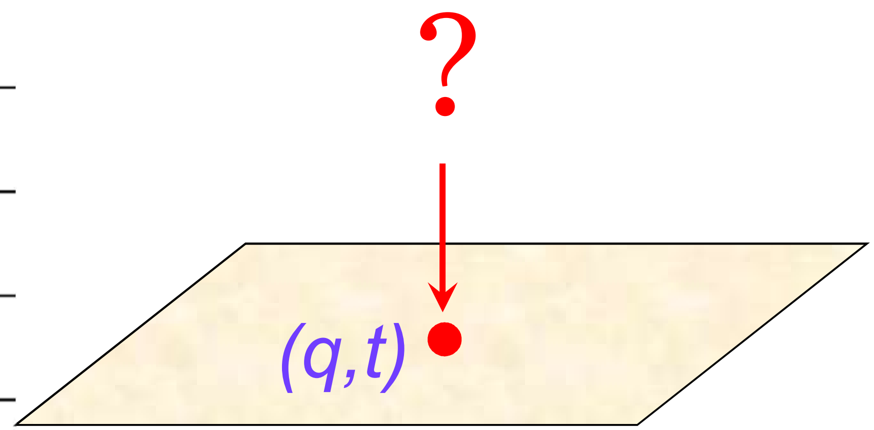
$$t^2 = -q^{-k}$$

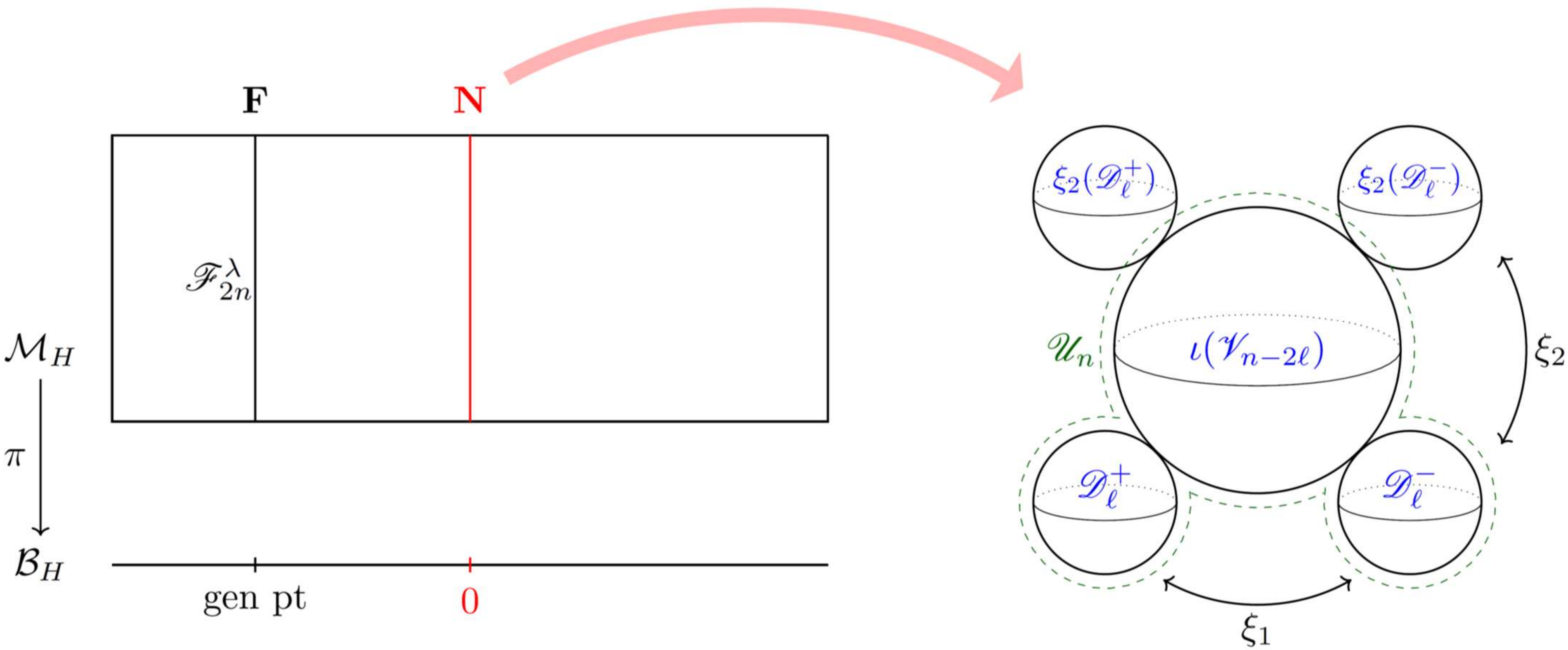
$$\mathscr{V}_{k+1} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$$

"additional series"
(Verlinde) representation



finite-dim rep	shortening condition
$\mathscr{F}_m$	$q^m = 1$
$\mathscr{U}_n$	$q^{2n} = 1$
$\mathscr{V}_{k+1}$	$t^2 = -q^{-k}$
$\mathscr{D}_\ell$	$t^2 = q^{-\ell+1/2}$





$$\mathrm{Sk}(M_3) = \frac{\mathbb{C}[q^{\pm\frac{1}{2}}](\text{isotopy classes of framed links in } M_3)}{\left(\quad = q^{-1/2} \right)\left(\; + q^{1/2} \quad , \quad = -q - q^{-1} \right)}$$

V.Turaev ('90)
J.Przytycki
:
S.Gunningham, D.Jordan, P.Safronov
T.Ekholm, V.Shende

$$\mathrm{Sk}(\Sigma) := \mathrm{Sk}(\Sigma \times [0,1])$$

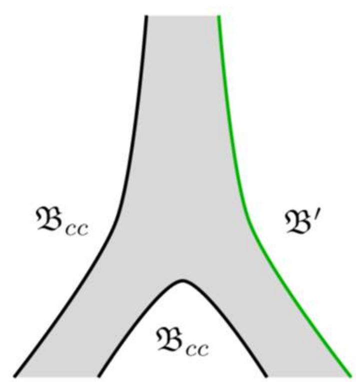
$\mathrm{Sk}(T^2)$ is a specialization of spherical DAHA

D.Bullock, J.Przytycki

stacking: $\mathrm{Sk}(\Sigma) \times \mathrm{Sk}(\Sigma) \to \mathrm{Sk}(\Sigma)$

V.Turaev ('91)

$$\begin{array}{ccc}
\mathrm{Sk}(\Sigma) & \cong & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}}) \\
\circlearrowright & & \circlearrowright \\
\mathrm{Sk}(M_3) & \cong & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{H})
\end{array}$$



S.G., P.Koroteev, S.Nawata, D.Pei, I.Saberi

Geometry

Quantum algebra

Quantization

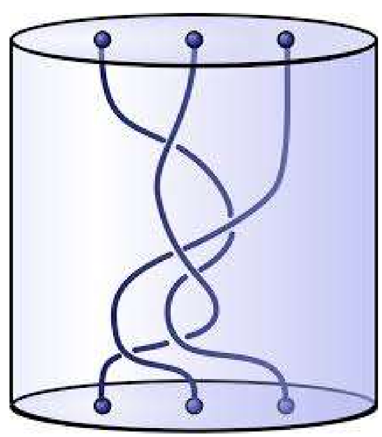
algebra $\mathcal{A}$, ...

A-model

X

MTC[X]

(modular) tensor category

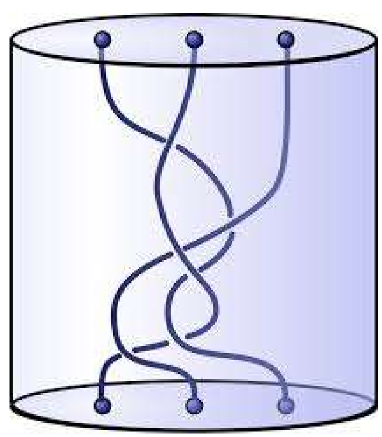


B-model

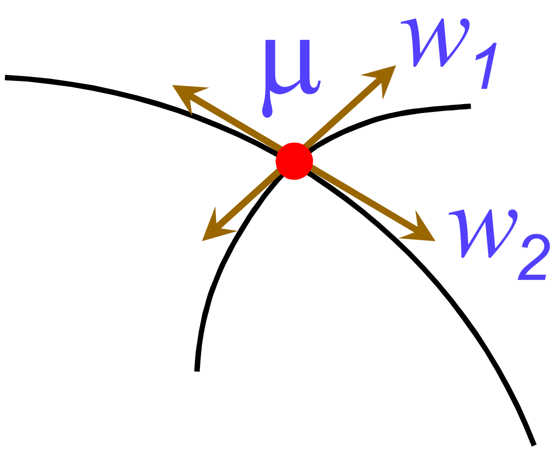
Example:

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \end{pmatrix}$$

$$T = \begin{pmatrix} e^{-\frac{\pi i}{15}} & 0 \\ 0 & e^{\frac{11\pi i}{15}} \end{pmatrix}$$



$U(1)_t \circlearrowright X =$ Coulomb branch, ...



$$T_{\lambda\lambda} = t^{\mu(\lambda)}$$

$$(S_{0\lambda})^2 = \frac{K_X^{1/2}}{\text{K-theory Euler class}(T_\lambda X)}$$

Example:

$$U(1)_t \curvearrowright X = \text{Coulomb branch, ...}$$

2 fixed points

$$\begin{array}{lcc} \mu = & 0 & 1/5 \\ w_1 = & 2/5 & 6/5 \\ w_2 = & 3/5 & -1/5 \end{array}$$

Fibonacci MTC

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \end{pmatrix}$$

$$T = \begin{pmatrix} e^{-\frac{\pi i}{15}} & 0 \\ 0 & e^{\frac{11\pi i}{15}} \end{pmatrix}$$

Example:

$$U(1)_x \times U(1)_t \circlearrowright X = T^*\mathbb{C}\mathbf{P}^1$$

tri-holomorphic

holomorphic

$$TX|_{p_1} = x + t/x \Rightarrow (S_{00})^2 = \frac{1}{(1-x)(1-t/x)}$$

$$TX|_{p_2} = x^{-1} + tx \Rightarrow (S_{01})^2 = \frac{1}{(1-x^{-1})(1-tx)}$$

Geometry

of $\longleftrightarrow$ MTC

Galois action?

Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

E. Witten

“Rozansky-Witten invariants via formal geometry”

“Rozansky-Witten invariants via Atiyah classes”

Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

E. Witten

answers. In general, the analysis by cutting and summing over physical states is likely to be quite subtle if X is not compact, roughly because there is a continuum of almost Q-invariant states starting at zero energy. In the presence of such a continuum, formal arguments claiming to show a reduction to the Q-cohomology are hazardous at best. But if X is compact, the spectrum is discrete, and one will get a quite straightforward formalism involving a sum over finitely many physical states.

(cf. eq. (5.8)). If X is non-compact, the continuous spectrum starting at zero energy obstructs a reduction to a description with a finite-dimensional space of physical states. We therefore consider only compact X, such as $X = \mathrm{K3}$, to obtain the surgery formulas.

Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

E. Witten

$$\mathcal{H}(\Sigma_g) = \bigoplus_{q=0}^{\dim_{\mathbb{C}} X} H^q_{\bar\partial}(X, (\wedge^* V)^{\otimes g})$$

$$= \begin{cases} \bigoplus_{l=0}^{2n} H^{0,l}(X)\,, & g=0 \quad (\Sigma_g = S^2) \\ \bigoplus_{l,m=0}^{2n} H^{l,m}(X)\,, & g=1 \quad (\Sigma_g = T^2) \\ \vdots & \end{cases}$$

$$Z(S^1 \times \Sigma_g) = \mathrm{sdim}\mathcal{H}(\Sigma_g)$$

$$= \sum_{\lambda} (S_{0\lambda})^{2-2g}$$

equivariant

infinite-dimensional

$$Z(S^1 \times \Sigma_g) \;=\; \mathrm{sdim}\mathcal{H}(\Sigma_g)$$

$$= \sum_\lambda (S_{0\lambda})^{2-2g}$$

finite sum

$$= \sum_n t^n \mathrm{sdim}\mathcal{H}_n(\Sigma_g)$$

G.Moore, N.Nekrasov, S.Shatashvili
C.Teleman, C.Woodward
A.Gerasimov, S.Shatashvili
S.G., D.Pei
:

Thanks for listening.

Questions?