Semi-classical Toeplitz operators and geometric quantization on CR manifolds and complex manifolds with boundary

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Set up

- $M := \{z \in M';\ \rho(z) < 0\}$: domain with smooth boundary X.
- M': complex manifold of dimension n.
- ρ: defining function of M with $|d\rho| = 1$ on $X := \partial M$.

Set up

- Suppose that M' admits a compact d-dimensional Lie group G action.
- The G-action is holomorphic, preserves the boundary X.
- $(\cdot\,|\,\cdot)_M$: L^2 inner product on $\mathcal{C}^\infty(\overline{M})$ induced by the given G-invariant Hermitian metric.

Set up

- $H^0(\overline{M}) := \mathrm{Ker}\,\overline{\partial} \subset L^2(M)$: the space of global L^2 holomorphic functions.
- $H^0(\overline{M})^G := \{u \in H^0(\overline{M});\ h^*u = u,\ \text{for any } h \in G\}$: G-invariant L^2 holomorphic functions.
- $B_G : L^2(M) \to H^0(\overline{M})^G$: the orthogonal projection with respect to $(\cdot\,|\,\cdot)_M$ (G-invariant Bergman projection).

Motivation

- The study of B_G is important in quantization on complex manifolds with boundary.
- Q: what is $B_G(x, y)$?
- Can we have Guillemin-Sternberg type result: $H^0(\overline{M})^G \cong H^0(\overline{M}_G)$, M_G: reduced space?

Example

- $M := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3;\ |z_1|^4 + |z_2|^2 + |z_3|^2 < 1\right\}$.
- M admits an S^1-action:

$$S^1 \times M \to M, \quad e^{i\theta} \cdot (z_1, z_2, z_3) = (e^{-i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3).$$

- $H^0(\overline{M})$ and $H^0(\overline{M})^{S_1}$ are infinite dimensional.

Moment map

- $\omega_0 := J(d\rho)$, J is the complex structure on T^*M'.
- The moment map associated to the form ω_0 is the map $\mu : M' \to \mathfrak{g}^*$ defined by

$$\langle \mu(x), \xi \rangle = \omega_0(\xi_{M'}(x)), \qquad x \in M', \quad \xi \in \mathfrak{g}. \tag{1}$$

- $\mathfrak{g}$: Lie algebra of G, $\xi_{M'}$: vector field on M' induced by ξ.
- $\mu_X := \mu|_X : X \to \mathfrak{g}^*$ be the associated moment map on the CR manifold X.

Moment map

We assume that

- 0 is a regular value of μ_X,
- G acts freely on $\mu^{-1}(0) \cap X$, $\mu^{-1}(0) \cap X \neq \emptyset$,
- the Levi form $\mathcal{L}_x$ is positive or negative near $\mu^{-1}(0) \cap X$.
- $\mathcal{L}_x = \partial\overline{\partial}\rho|_{T^{1,0}X}$, $T^{1,0}X := T^{1,0}M' \cap \mathbb{C}TX$.

G-invariant Bergman kernel asymptotics, joint with Huang, Li and Shao

Theorem 0

- *Let* $\tau \in \mathcal{C}^\infty(\overline{M})$ *with* $\operatorname{supp}\tau \cap \mu^{-1}(0) \cap X = \emptyset$.
- $\tau B_G \equiv 0 \mod \mathcal{C}^\infty(\overline{M} \times \overline{M})$, $B_G \tau \equiv 0 \mod \mathcal{C}^\infty(\overline{M} \times \overline{M})$.

G-invariant Bergman kernel asymptotics, joint with Huang, Li and Shao

Theorem 0

- *Let $p \in \mu^{-1}(0) \cap X$. Let U be an open local coordinate patch of p in M', $D := U \cap X$.*
- *If Levi form is negative on D, then*

$$B_G(z, w) \equiv 0 \mod \mathcal{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \quad (2)$$

G-invariant Bergman kernel asymptotics, joint with Huang, Li and Shao

Theorem 0

- *If Levi form is positive on D, then*

$$B_G(z,w) \equiv \int_0^{+\infty} e^{it\Psi(z,w)} b(z,w,t)dt \qquad (3)$$
$$\mod \mathcal{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

- $b(z,w,t) \sim \sum_{j=0}^{+\infty} t^{n-\frac{d}{2}-j} b_j(z,w)$ *in* $S^{n-\frac{d}{2}}_{1,0}(((U\times U)\cap(\overline{M}\times\overline{M}))\times\mathbb{R}_+)$.
- $b_0(x,x)\neq 0$, *for every* $x\in\mu^{-1}(0)\cap D$.

G-invariant Bergman kernel asymptotics, joint with Huang, Li and Shao

Theorem 0

- $\Psi(z,w) \in \mathcal{C}^\infty(((U \times U) \cap (\overline{M} \times \overline{M}))$, $\operatorname{Im}\Psi \geq 0$.
- $\Psi(z,z) = 0$, $z \in \mu^{-1}(0) \cap D$.
- $\operatorname{Im}\Psi(z,w) > 0$ *if* $(z,w) \notin \operatorname{diag}\left((\mu^{-1}(0) \cap D) \times (\mu^{-1}(0) \cap D)\right)$.
- $d_x\Psi(x,x) = -\omega_0(x) - id\rho(x)$, $d_y\Psi(x,x) = \omega_0(x) - id\rho(x)$, $x \in \mu^{-1}(0) \cap D$.

G-invariant Bergman kernel asymptotics, joint with Huang, Li and Shao

Theorem 0

- $B_G(z,w) = \frac{F(z,w)}{(-i\Psi(z,w))^{n-\frac{d}{2}+1}} + G(z,w)\log(-i\Psi(z,w))$.
- $F(z,w), G(z,w) \in \mathcal{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Remark

- We show that B_G is a complex Fourier integral operator near positive part of $\mu^{-1}(0) \cap X$.
- When G is trivial and X is strongly pseudoconvex, Fefferman(1974) established an asymptotic expansion for $B^G = B$ at the diagonal.
- A full asymptotic expansion of B was obtained by Boutet de Monvel and Sjöstrand(1976).

Remark

- The asymptotic of B plays an important role in some important problems in several complex variables.
- By using Theorem 0 , we get G-invariant version of Fefferman's result about regularity of biholomorphic maps on strongly pseudoconvex domains of $\mathbb{C}^n$.

Geometric quantization on complex manifolds with boundary

- $\mu_X^{-1}(0)$: d-codimensional submanifold of X.
- $\mu^{-1}(0) \cap X = \widehat{X} \cup \widetilde{X}$, $\widehat{X}$: strongly pseudoconvex, $\widetilde{X}$: strongly pseudoconcave.
- $\widehat{X}_G := \widehat{X}/G$, $\widetilde{X}_G = \widetilde{X}/G$.
- Fact: $\widehat{X}_G$ is a strongly pseudoconvex CR manifold.

Guillemin-Sternberg map

- $H^0_b(\widehat{X}_G)_s := \left\{u \in W^s(\widehat{X}_G);\ \overline{\partial}_b u = 0\right\}$.
- $\overline{\partial}_b$: the tangential Cauchy-Riemann operator on $\widehat{X}$.
- $H^0(\overline{M})^G_s := \{u \in W^s(\overline{M});\ \overline{\partial} u, h^* u = u,\ \ \forall h \in G\}$.

- Guillemin-Sternberg map:

$$\tilde{\sigma}_G : H^0(\overline{M})^G_s \to H^0_b(\widehat{X}_G)_{s-\frac{d}{4}-\frac{1}{2}}, \qquad u \to \iota_{G,\widehat{X}} \circ \iota^*_{\widehat{X}} \circ \gamma \circ u. \tag{4}$$

- $\iota_{\widehat{X}} : \widehat{X} \hookrightarrow X$: natural inclusion.
- $\iota_{G,\widehat{X}} : \mathcal{C}^\infty(\widehat{X})^G \to \mathcal{C}^\infty(\widehat{X}_G)$: natural identification.
- γ: the operator of the restriction to the boundary X.

Geometric quantization on complex manifolds with boundary

Theorem I (joint with Huang, Li and Shao)

- *For every* $s \in \mathbb{R}$, *the Guillemin-Sternberg map* (4) *is Fredholm.*
- $\mathrm{Ker}\,\tilde{\sigma}_{G,s}$ *and* $\mathrm{Coker}\,\tilde{\sigma}_{G,s}$ *are finite dimensional subspaces of* $H^0(\overline{M})^G \cap \mathcal{C}^\infty(\overline{M})^G$ *and* $H^0_b(\widehat{X}_G) \cap \mathcal{C}^\infty(\widehat{X}_G)$ *respectively.*
- $\mathrm{Ker}\,\tilde{\sigma}_{G,s}$ *and* $\mathrm{Coker}\,\tilde{\sigma}_{G,s}$ *are independent of* s.

Remark

- This result can be used to construct global G-invariant holomorphic functions on M with given singularities at the boundary.

Theorem II (joint with Huang, Li and Shao)

- *Assume that* 0 *is a regular value of* μ, G *acts freely on* $\mu^{-1}(0)$.
- $M'_G := \mu^{-1}(0)/G$, $M_G := (\mu^{-1}(0) \cap M)/G$.
- M_G *is a complex manifold in* M'_G *with smooth boundary* X_G.

Geometric quantization on complex manifolds with boundary

Theorem II (joint with Huang, Li and Shao)

- $\sigma_G = \sigma_{G,s} : H^0(\overline{M})^G_s \to H^0(\overline{M}_G)_{s-\frac{d}{4}}$: *holomorphic Guillemin-Sternberg map.*
- *The holomorphic Guillemin-Sternberg map is Fredholm.*
- $\operatorname{Ker} \sigma_{G,s}$ *and* $\mathrm{Coker}\, \sigma_{G,s}$ *are finite dimensional subspaces of* $H^0(\overline{M})^G \cap \mathcal{C}^\infty(\overline{M})^G$ *and* $H^0(\overline{M}_G) \cap \mathcal{C}^\infty(\overline{M}_G)$ *respectively.*
- $\operatorname{Ker} \sigma_{G,s}$ *and* $\mathrm{Coker}\, \sigma_{G,s}$ *are independent of* s.

Examples and applications

- $(L, h^L) \to Y$: holomorphic line bundle over a compact complex manifold Y.
- $M = \left\{v \in L^*; \, |v|^2_{h^{L^*}} < 1\right\}$, $M' = L^*$.
- $X = \partial M = \left\{v \in L^*; \, |v|^2_{h^{L^*}} = 1\right\}$: circle bundle.
- M' admits a natural S^1-action (acting on fiber).
- Assume that G commutes with S^1 (for example, G acts on base manifold Y).

Examples and applications

- $H^0_k(\overline{M})^G = \left\{u \in H^0(\overline{M})^G;\, (e^{i\theta})^* u = e^{ik\theta} u\right\}$.
- $H^0_{b,k}(X_G) = \left\{u \in H^0_b(X_G);\, (e^{i\theta})^* u = e^{ik\theta} u\right\}$.
- From Theorem I, for $|k| \gg 1$,

$$H^0_k(\overline{M})^G \cong H^0_{b,k}(X_G) \cong H^0(Y_G, L^k_G). \tag{5}$$

- Consider $M = \left\{ v \in L^*;\ \frac{1}{2} < |v|^2_{h^{L^*}} < 1 \right\}$.
- From Theorem II, for $|k| \gg 1$,

$$H^0_k(\overline{M})^G \cong H^0_k(\overline{M}_G). \tag{6}$$

Examples and applications

- We can generalize (5) and (6) to general domain M with a holomorphic compact Lie group action H such that H commutes with G.
- $k \leftrightarrow$ irreducible representation of H.

Toeplitz operator view point

- $(L, h^L) \to Y$: holomorphic line bundle over a compact complex manifold Y.
- R: vector field on M' induced by the S^1 action of L^*.
- G-invariant Toeplitz operator: $T_R^G := B_G \circ (-iR) \circ B_G$, $B_G : L^2(M) \to H^0(\overline{M})^G$: G-invariant Bergman projection.
- Toeplitz operator on X_G: $T_{R_{X_G}} := S_{X_G} \circ (-iR_{X_G}) \circ S_{X_G}$, $S_{X_G} : L^2(X_G) \to \mathrm{Ker}\,\overline{\partial}_b$: Szegő projection.

Toeplitz operator view point

- $H^0_k(\overline{M})^G = \{u \in L^2(M);\ T^G_R u = ku\}$.
- $H^0_{b,k}(X_G) = \Big\{u \in L^2(X_G);\ T_{R_{X_G}} u = ku\Big\}$.
- For $|k| \gg 1$, $E_k(T^G_R) \cong E_k(T_{R_{X_G}})$.
- $E_k(T^G_R)(E_k(T_{R_{X_G}}))$ eigenspace of $T^G_R(T_{R_{X_G}})$ corresponding to the eigenvalue k.
- The eigenvalues of T^G_R are not integer in general.

- Assume that X is strongly pseudoconvex.
- Fix a G-invariant Reeb vector field T on X, that is $T \in \mathcal{C}^\infty(X, TX)$, $\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T$ (we can take $T = J(\frac{\partial}{\partial\rho})|_X$).
- R: G-invariant self-adjoint vector field on M' so that $R = \frac{1}{2}((-iT) + (-iT)^*)$ on X.
- We can define Toeplitz operators T_R^G, $T_{R_{X_G}}$ as above.

- Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}_+)$.
- Let $\chi_k(T^G_R) := \chi(k^{-1}T^G_R)$, $\chi_k(T_{R_{X_G}}) := \chi(k^{-1}T_{R_{X_G}})$.
- $\chi_k(T^G_R)(\chi_k(T_{R_{X_G}}))$: functional calculus of $k^{-1}T^G_R(k^{-1}T_{R_{X_G}})$ with respect to χ.

Theorem III (joint with Herrmann, Marinescu and Shen)

- $\chi_k(T_{R_{X_G}})(x,y) = \int e^{ikt\varphi(x,y)}a(x,y,t,k)dt + O(k^{-\infty})$, $k \gg 1$,
- $a(x,y,t,k) \sim \sum_{j=0}^{+\infty} k^{n-d-j}a_j(x,y,t)$,
- $\mathrm{Supp}\,_t a(x,y,t,k), \mathrm{Supp}\,_t a_j(x,y,t,k) \subset \mathrm{Supp}\,\chi$,
- $a_0(x,x,t) = \frac{1}{2\pi^{n-d}}\chi(t)t^n \det \mathcal{L}_{X_G,x}$,
- $\mathrm{Im}\,\varphi \geq 0$, $\varphi(x,y) = 0$ *if and only if* $x = y$.

G-invariant Toeplitz operators asymptotics on complex manifolds with boundary

Theorem IV (in preparation)

- *U: open set in M'.*
- *If $U \cap \mu^{-1}(0) \cap X = \emptyset$.*
- $\chi_k(T_R^G)(x,y) \equiv 0 \mod O(k^{-\infty})$ *on* $(U \times U) \cap (\overline{M} \times \overline{M})$.
- *If $U \cap \mu^{-1}(0) \cap X \neq \emptyset$.*
- $\chi_k(T_R^G)(x,y) \equiv \int e^{ik\Psi(x,y,t)} b(x,y,t,k)dt \mod O(k^{-\infty})$ *on* $(U \times U) \cap (\overline{M} \times \overline{M})$.

G-invariant Toeplitz operators asymptotics on complex manifolds with boundary

Theorem IV (in preparation)

- $b(x,y,t,k) \sim \sum_{j=0}^{+\infty} k^{n+1-\frac{d}{2}-j} b_j(x,y,t)$,
- $\mathrm{Supp}\,_t b(x,y,t,k), \mathrm{Supp}\,_t b_j(x,y,t,k) \subset \mathrm{Supp}\,\chi$, $b(x,y,t), b_j(x,y,t) \in \mathcal{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})\times\mathbb{R}_+)$, $b_0(x,x,t) \neq 0$.
- $\mathrm{Im}\,\Psi \geq 0$, $\Psi(x,y) \geq C\Big((\mathrm{dist}\,(x,\mu^{-1}(0)\cap X))^2 + (\mathrm{dist}\,(y,\mu^{-1}(0)\cap X))^2\Big)$.
- $\Psi = 0$ *if and only if* $x = y \in \mu^{-1}(0)\cap X$.

Quantization commutes with reduction for Toeplitz operators on complex manifolds with boundary

Theorem V

- *Fix* $0 < \delta_1 < \delta_2$.
- *We have for* $k \gg 1$,
- $\oplus_{\lambda \in [k\delta_1, k\delta_2]} E_\lambda(T_R^G) \cong \oplus_{\lambda \in [k\delta_1, k\delta_2]} E_\lambda(T_{R_{X_G}})$.

Application

Theorem VI

- $\dim \oplus_{\lambda\in[k\delta_1,k\delta_2]} E_\lambda(T_R^G) = \frac{k^{n-d}}{2\pi^{n-d}}\int_{X_G}\int_{\delta_1}^{\delta_2} t^{n-d-1}\det \mathcal{L}_{X_G,x}\,dt\,dV_{X_G} + O(k^{n-d-1})$ *(G-invariant Boutet de Monvel-Guillemin Weyl law for domains).*
- *We can replace R to a pseudodiffernetial operator.*

Example

- $M := \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n;\ |z_1|^4 + |z_2|^2 + \cdots + |z_n|^2 < 1 \right\}$.
- M admits an $G = S^1$-action:

$$S^1 \times M \to M, \quad e^{i\theta} \cdot (z_1, z_2, \ldots, z_n) = (e^{-i\theta} z_1, e^{i\theta} z_2, \ldots, e^{i\theta} z_n).$$

- M is a weakly pseudoconvex domain in $\mathbb{C}^n$.

Example

- 0 is a regular value of μ_X,
- G acts freely on $\mu^{-1}(0) \cap X$, $\mu^{-1}(0) \cap X \neq \emptyset$,
- the Levi form $\mathcal{L}_x$ is positive near $\mu^{-1}(0) \cap X$.

Example

- $$H^0(\overline{M})^G = \operatorname{span}\{z_1^{\alpha_1}\cdots z_n^{\alpha_n};\\ -\alpha_1+\alpha_2+\cdots+\alpha_n=0, (\alpha_1,\ldots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n\}.$$
- $X_G = \left\{(z_2,\ldots,z_n)\in\mathbb{C}^{n-1};\ |z_2|^2+\cdots+|z_n|^2=\frac{2}{3}\right\}.$
- $H_b^0(X_G) = \operatorname{span}\{z_2^{\alpha_2}\cdots z_n^{\alpha_n}|_{X_G}; (\alpha_2,\ldots,\alpha_n)\in(\mathbb{N}\cup\{0\})^{n-1}\}.$
- $H^0(\overline{M})^G \cong H_b^0(X_G).$

Example

- $0 \in \mathbb{C}^n$ is not a regular value of the moment map.
- Consider $M := \left\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n;\ \frac{1}{2} < |z_1|^4 + |z_2|^2 + \cdots + |z_n|^2 < 1\right\}$.
- $M_G = \left\{(z_2, \ldots, z_n) \in \mathbb{C}^{n-1};\ \frac{1}{3} < |z_2|^2 + \cdots + |z_n|^2 < \frac{2}{3}\right\}$.
- $H^0(\overline{M})^G \cong H^0(\overline{M}_G)$.

Example

- Let $R = \sum_{j=1}^{n}(i\beta_j z_j \frac{\partial}{\partial z_j} - i\beta_j \overline{z}_j \frac{\partial}{\partial \overline{z}_j})$, $(\beta_1, \ldots, \beta_n) \in \mathbb{R}_+^n$.
- $R|_X$ is a Reeb vector field.
- $$\text{Eigenvalues of } T_R^G$$
  $$= \{\lambda = m_1\beta_1 + \cdots + m_n\beta_n;$$
  $$-m_1 + m_2 + \cdots + m_n = 0, (m_1, \ldots, m_n) \in (\mathbb{N} \cup \{0\})^n\}.$$

Example

- $$E_\lambda(T_R^G) \\ = \mathrm{span}\,\{z_1^{m_1}\cdots z_n^{m_n}; \\ \quad -m_1+m_2+\cdots+m_n=0, (m_1,\ldots,m_n)\in(\mathbb{N}\cup\{0\})^n, \\ \beta_1 m_1+\cdots+\beta_n m_n=\lambda\}.$$

Example

- From Theorem VI, we have
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$$\dim \oplus_{\lambda\in[k\delta_1,k\delta_2]} E_\lambda(T_R^G)$$
$$= |\{(m_1,\ldots,m_n)\in(\mathbb{N}\cup\{0\})^n;$$
$$k\delta_1\leq\beta_1 m_1+\cdots+\beta_n m_n\leq k\delta_2, -m_1+m_2+\cdots+m_n=0\}|$$
$$=\frac{k^{n-1}}{2\pi^{n-1}}\int_{X_G}\int_{\delta_1}^{\delta_2} t^{n-2}\det\mathcal{L}_{X_G,x}dtdV_{X_G}+O(k^{n-2}).$$